Relativistic Shocks

Relativistic Jumping Conditions

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HELMHOLTZ WEIZMANN
RESEARCH SCHOOL
MULTIMESSENGER ASTRONOMY

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

DESY.

Relativistic?

- Be careful:

1. relativistic shock = shock moving at relativistic speeds

   → $(\beta\Gamma)_{\text{shock}} > 1$

2. relativistic equation of state: particles in the fluid move at relativistic speeds

   → $\langle\beta\Gamma\rangle_{\text{particle}} > 1$

Strong shocks

- as in the non-relativistic case a shock is said to be strong if:

  Mach number $\mathcal{M} = \frac{\beta_u \Gamma_u}{\beta_{s,u} \Gamma_{s,u}} > 1$

  sound upstream

  → then $\mathcal{M}^2 = \frac{\text{ram pressure}}{\text{thermal pressure}}$

- equivalent to cold upstream medium: $\beta_s^2 \sim \frac{p_u}{\rho_u} \ll 1$
- both depending only on upstream frame (*initial condition*)

Non-relativistic jumping conditions

mass

$$\rho_d v_d = \rho_u v_u$$

$$\xrightarrow{\rho = m_p n}$$

number

$$n_d \beta_d = n_u \beta_u$$

momentum

$$\rho_d v_d^2 + p_d = \rho_u v_u^2 + p_u$$

energy

$$v_d \left[\frac{1}{2} \rho_d v_d^2 + \frac{\hat{\gamma}}{\hat{\gamma} - 1} p_d \right] = v_u \left[\frac{1}{2} \rho_u v_u^2 + \frac{\hat{\gamma}}{\hat{\gamma} - 1} p_u \right]$$

Non-relativistic jumping conditions

mass

$$\rho_d v_d = \rho_u v_u$$

$$\xrightarrow{\rho = m_p n}$$

number

$$n_d \beta_d = n_u \beta_u$$

$$\Rightarrow \Gamma$$

momentum

$$\rho_d v_d^2 + p_d = \rho_u v_u^2 + p_u$$

$$\Rightarrow \Gamma^2$$

energy

$$v_d \left[\frac{1}{2} \rho_d v_d^2 + \frac{\hat{\gamma}}{\hat{\gamma} - 1} p_d \right] = v_u \left[\frac{1}{2} \rho_u v_u^2 + \frac{\hat{\gamma}}{\hat{\gamma} - 1} p_u \right]$$

$$\Rightarrow \Gamma^2$$

Number conservation

- in rest frame: n
- define number flux 4 vector: $N^{\mu} = nu^{\mu} = n\left(\Gamma, \Gamma v^{i}\right)$
- conservation of number of particles: $\boxed{\partial_{\mu} N^{\mu} = 0}$

$$\partial_{\mu} N^{\mu} = \frac{1}{c} \partial_{t}(n\Gamma) + \partial_{i}\left(n\Gamma\beta^{i}\right) = 0$$

- relativistic version of continuity equation ($\beta \ll 1$: $\partial_{t} n + \nabla(n\vec{v}) = 0$)
  (in rest frame no flux and $\partial_{t} n = 0$)

Energy, enthalpy,... - density

- in relativistic physics new concept of rest mass
- sum up all internal/isotropic “energy reservoirs” of a fluid: **enthalpy**

internal degrees of freedom

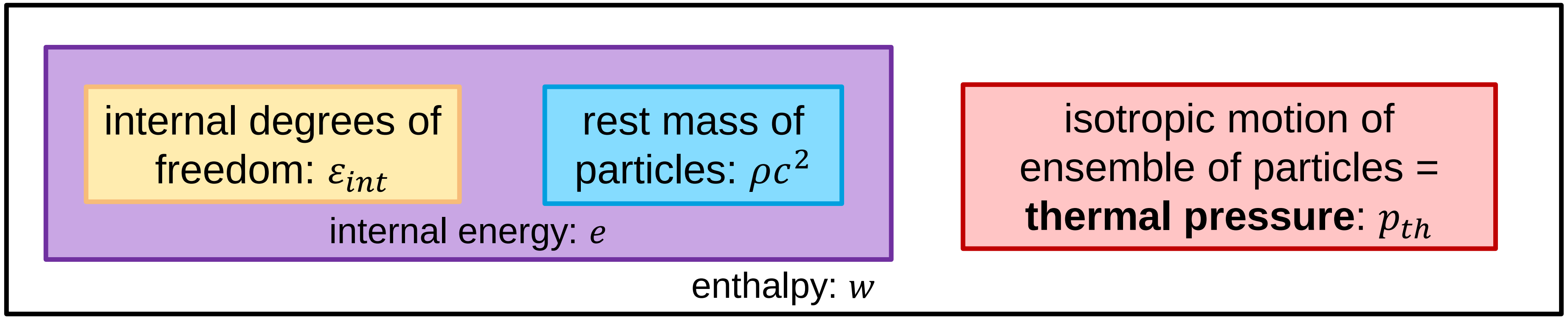
rest mass density

thermal pressure (also an energy density)

$$\rightarrow w = \underbrace{\varepsilon_{\mathrm{int}} + \rho c^2}_{\text{(internal) energy density } e} + p_{\mathrm{th}}$$

different from non-relativistic definition:

$$w = \varepsilon_{\mathrm{int}} + p_{\mathrm{th}}$$



internal degrees of freedom: ε_{int}

rest mass of particles: ρc^2

internal energy: e

isotropic motion of ensemble of particles = **thermal pressure**: p_{th}

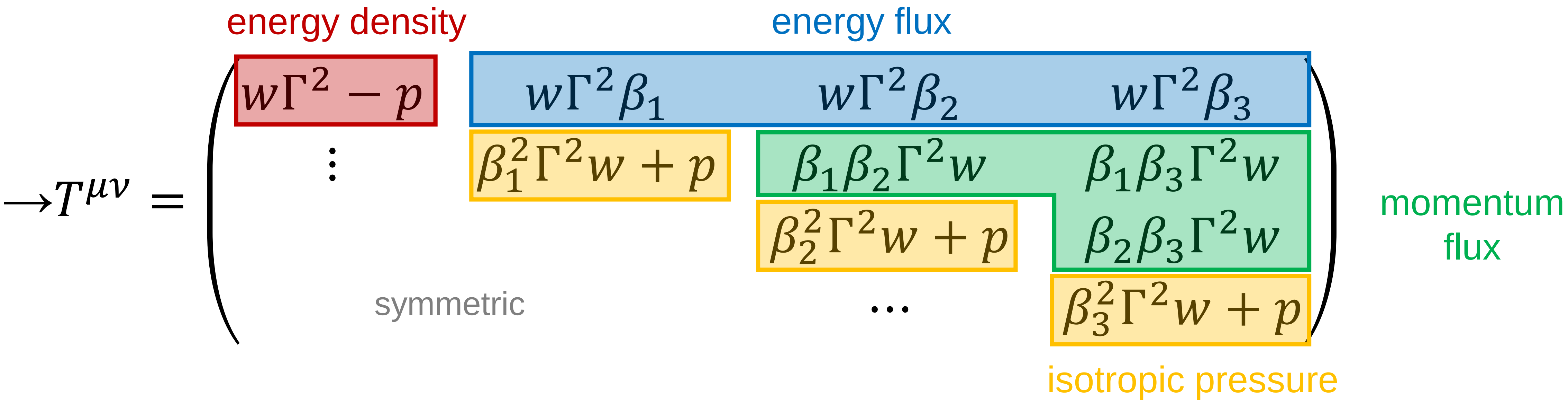
enthalpy: w

Conservation of energy and momentum

- energy and momentum are a combined concept in relativity

  → **energy momentum tensor** $\boldsymbol{T^{\mu\nu} = wu^{\mu}u^{\nu} + p_{\mathrm{th}}g^{\mu\nu}}$

- perfect fluid: no viscosity/heat conduction



energy density

energy flux

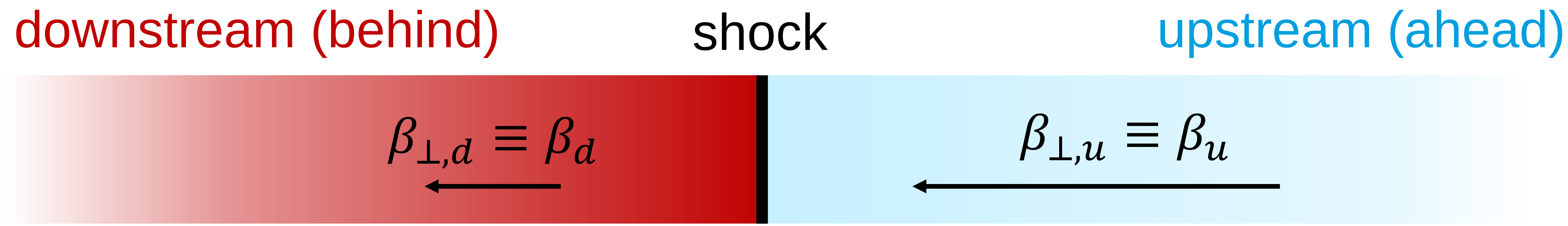
momentum flux

isotropic pressure

$$\rightarrow T^{\mu\nu} = \begin{pmatrix} w\Gamma^2 - p & w\Gamma^2\beta_1 & w\Gamma^2\beta_2 & w\Gamma^2\beta_3 \\ \vdots & \beta_1^2\Gamma^2 w + p & \beta_1\beta_2\Gamma^2 w & \beta_1\beta_3\Gamma^2 w \\ & & \beta_2^2\Gamma^2 w + p & \beta_2\beta_3\Gamma^2 w \\ \text{symmetric} & & \cdots & \beta_3^2\Gamma^2 w + p \end{pmatrix}$$

- 4 equations (1 energy + 3 mom.): $\boxed{\partial_\mu T^{\mu\nu} = 0}$

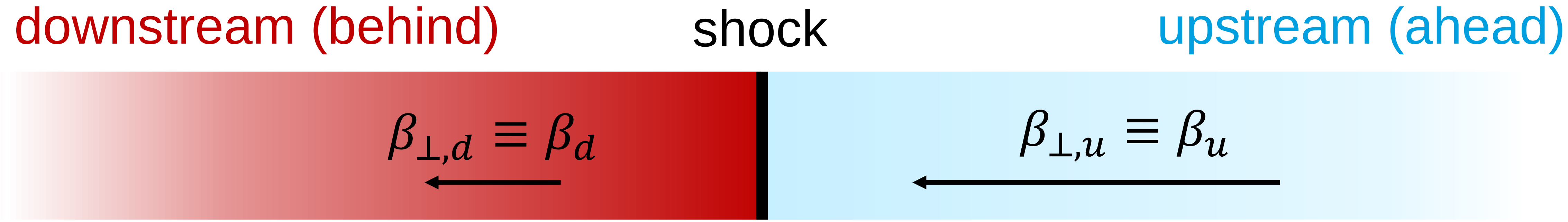
Shock rest frame and 1D



$$N^{\mu} = (n\Gamma, n\beta\Gamma, 0\,, 0)$$

$$T^{\mu\nu} = \begin{pmatrix} w\Gamma^2 - p & w\Gamma^2\beta & 0 & 0 \\ w\Gamma^2\beta & \beta^2\Gamma^2 w + p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Conservation across shock



- assume conservation of particles/4-momentum across the shock

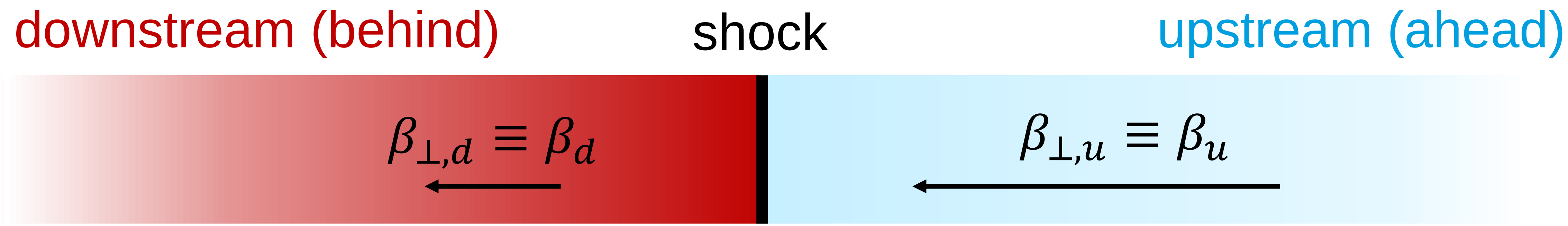
→ conservation of fluxes across shock $\partial_\mu X^\mu = \frac{1}{c}\partial_t X^0 - \partial_i X^i = 0$

→ no sources ($\frac{1}{\mathrm{c}}\partial_t X^0 = 0$): → $\partial_i X^i = 0$

→integrate across infinitesimal small box $\lim_{\epsilon \to 0} \int_{-\epsilon/2}^{\epsilon/2} \partial_i X^i = X_u - X_d = 0$

→$X^i_{\mathrm{upstream}} = X^i_{\mathrm{downstream}}$ ⇔ $\boxed{[\![X^i]\!] = 0}$

Relativistic jumping conditions



downstream (behind)

shock

upstream (ahead)

$\beta_{\perp,d} \equiv \beta_d$

$\beta_{\perp,u} \equiv \beta_u$

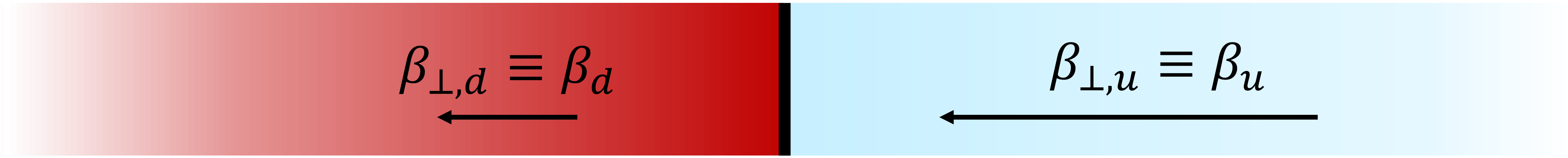
$$N^{\mu} = (n\Gamma, n\beta\Gamma, 0\,, 0)$$

$$T^{\mu\nu} = \begin{pmatrix} w\Gamma^2 - p & w\Gamma^2\beta & 0 & 0 \\ w\Gamma^2\beta & \beta^2\Gamma^2 w + p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

DESY.

Relativistic jumping conditions

downstream (behind) shock upstream (ahead)

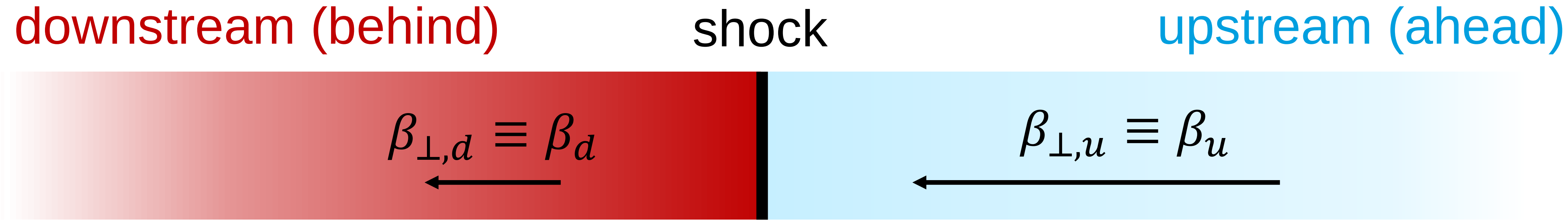


$$N^\mu = (\underbrace{n\Gamma,}_{\partial_t}\underbrace{n\beta\Gamma,}_{\partial_x} 0\,, 0)$$

$$\boxed{[\![n\Gamma\beta]\!] = 0}$$

$$T^{\mu\nu} = \begin{pmatrix} w\Gamma^2 - p & w\Gamma^2\beta & 0 & 0 \\ w\Gamma^2\beta & \beta^2\Gamma^2 w + p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Relativistic jumping conditions



downstream (behind)

shock

upstream (ahead)

$\beta_{\perp,d} \equiv \beta_d$

$\beta_{\perp,u} \equiv \beta_u$

$$N^\mu = (\underset{\partial_t}{n\Gamma}, \underset{\partial_x}{n\beta\Gamma}, 0\,, 0)$$

$$T^{\mu\nu} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccc} \partial_t & \partial_x & & \\ \end{array}$$

$$T^{\mu\nu} = \begin{pmatrix} w\Gamma^2 - p & w\Gamma^2\beta & 0 & 0 \\ w\Gamma^2\beta & \beta^2\Gamma^2 w + p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$\boxed{[\![n\Gamma\beta]\!] = 0}$$

$$\boxed{[\![w\Gamma^2\beta]\!] = 0}$$

$$\boxed{[\![\beta^2\Gamma^2 w + p]\!] = 0}$$

Non-relativistic jumping conditions

mass

$$\rho_d v_d = \rho_u v_u$$

number

$$n_d \beta_d = n_u \beta_u$$

momentum

$$\rho_d v_d^2 + p_d = \rho_u v_u^2 + p_u$$

energy

$$v_d \left[\frac{1}{2} \rho_d v_d^2 + \frac{\hat{\gamma}}{\hat{\gamma} - 1} p_d \right] = v_u \left[\frac{1}{2} \rho_u v_u^2 + \frac{\hat{\gamma}}{\hat{\gamma} - 1} p_u \right]$$

Transition of jumping conditions

mass

$$[\![\rho v]\!] = 0$$

number

$$[\![n\beta]\!] = 0$$

momentum

$$[\![\rho v^2 + p]\!] = 0$$

energy

$$\left[\!\!\left[v\left(\frac{1}{2}\rho v^2 + \frac{\hat{\gamma}}{\hat{\gamma}-1}p\right) \right]\!\!\right] = 0$$

Transition of jumping conditions

mass

$$[\![\rho v]\!] = 0$$

$$\longrightarrow$$

number

$$[\![n\beta]\!] = 0$$

$$\xrightarrow{\Gamma}$$

number density flux

$$\boxed{[\![n\Gamma\beta]\!] = 0}$$

momentum

$$[\![\rho v^2 + p]\!] = 0$$

$$\xrightarrow{\Gamma^2}$$

momentum density flux

$$\boxed{[\![\beta^2\Gamma^2 w + p]\!] = 0}$$

energy

$$\left[\!\!\left[v\left(\frac{1}{2}\rho v^2 + \frac{\hat{\gamma}}{\hat{\gamma}-1}p\right)\right]\!\!\right] = 0$$

$$\xrightarrow{\Gamma^2}$$

enthalpy density flux

$$\boxed{[\![w\Gamma^2\beta]\!] = 0}$$

Equation of state

- Jumping conditions = system of equations

  → relates 3 upstream variables to 3 downstream variables, e.g. $\{\beta, n, w\}$ or $\{\beta, n\ , p\}$

  → however equations contain further degree of freedom

  → need additional relation: **equation of state**

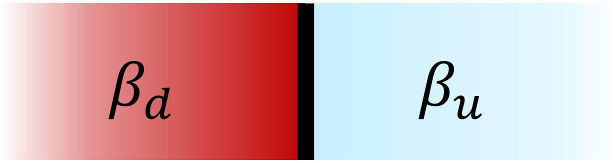
  → relates thermal pressure p_{th} to internal energy e

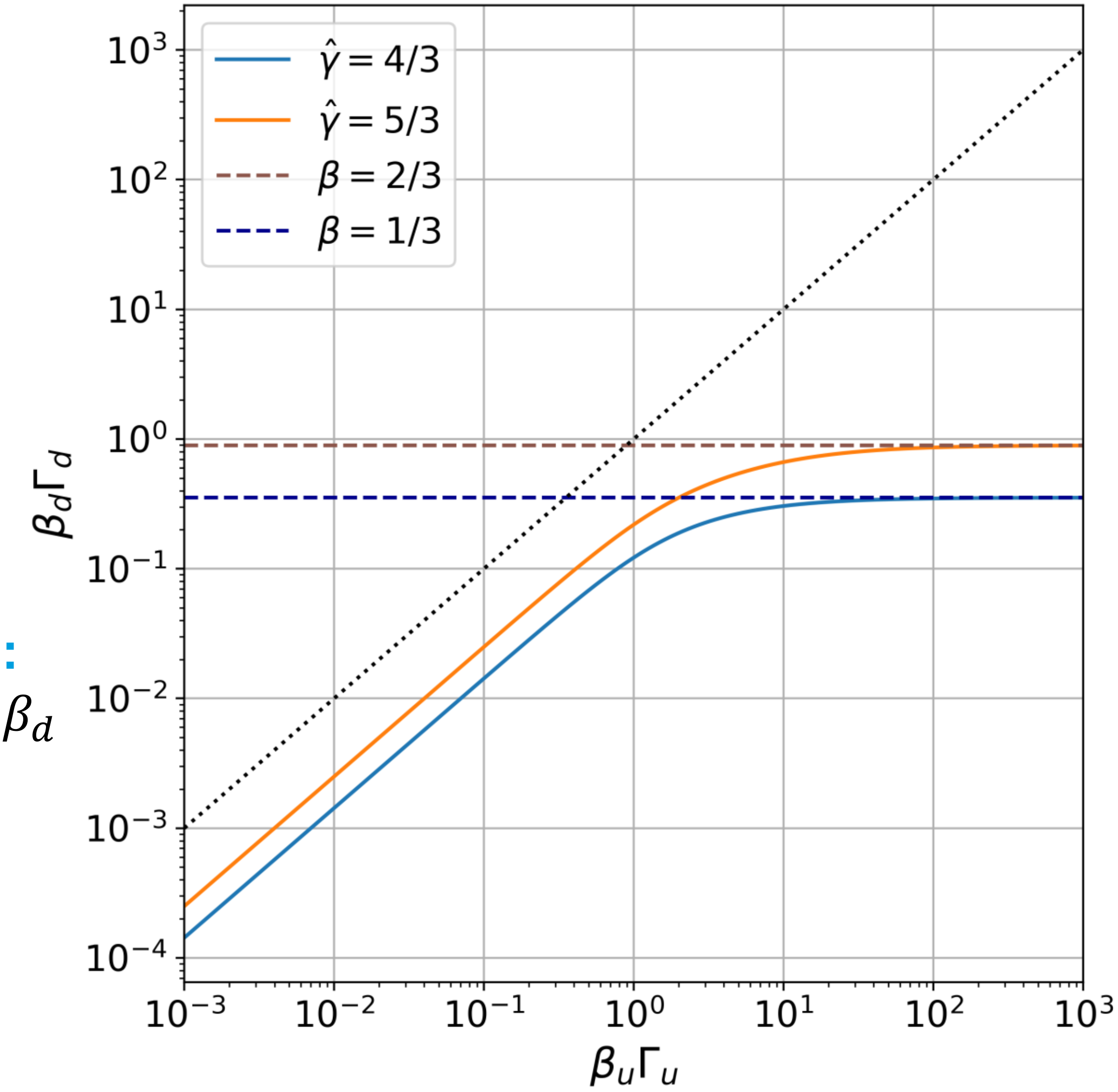
- polytropic equation of state: $\boxed{p = (\hat{\gamma} - 1)(e - \rho)}$

  → $e = \frac{p}{\hat{\gamma}-1} + \rho$

  → $\boxed{w = e + p = \frac{\hat{\gamma}}{\hat{\gamma}-1} p + \rho}$

Deceleration of fluid at a strong shock

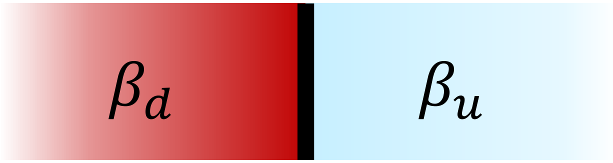


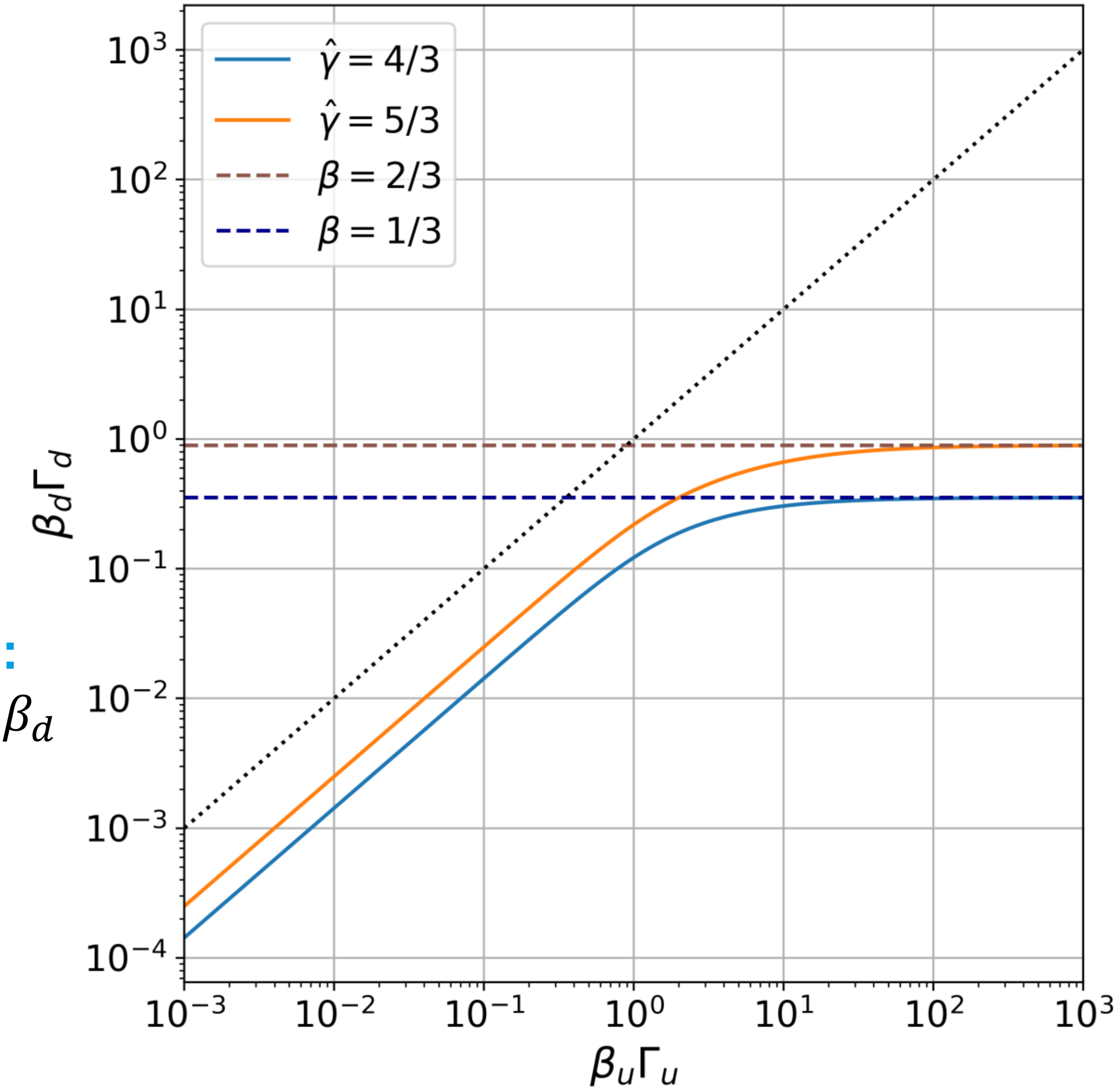


non-relativistic regime:

constant ratio of β_u and β_d
→ constant deceleration efficiency

Deceleration of fluid at a strong shock





$$\beta_u \to 1$$

$$\beta_d \to \hat{\gamma} - 1 = \begin{cases} \frac{2}{3} & \text{for } \hat{\gamma} = 5/3 \\ \frac{1}{3} & \text{for } \hat{\gamma} = 4/3 \end{cases}$$

non-relativistic regime:

constant ratio of β_u and β_d

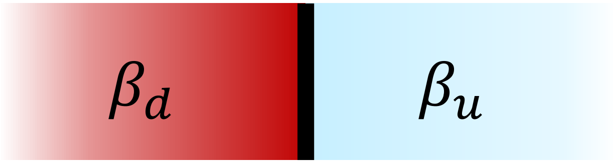
→ constant deceleration efficiency

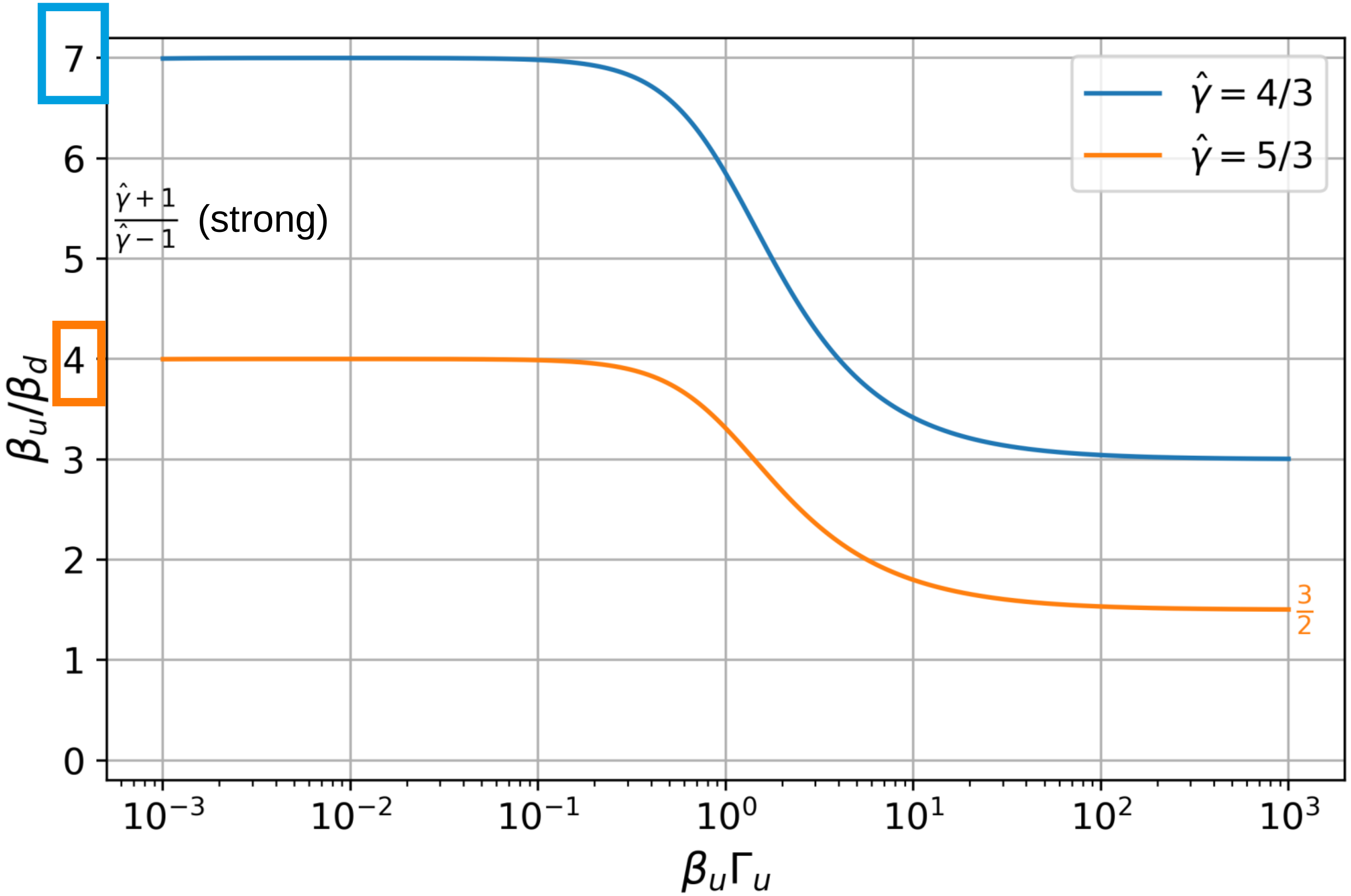
relativistic regime:

constant value of β_d

→ shock always decelerates downstream flow to non-relativistic speeds

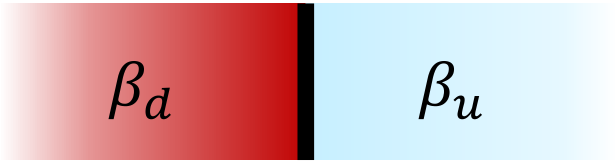
Deceleration

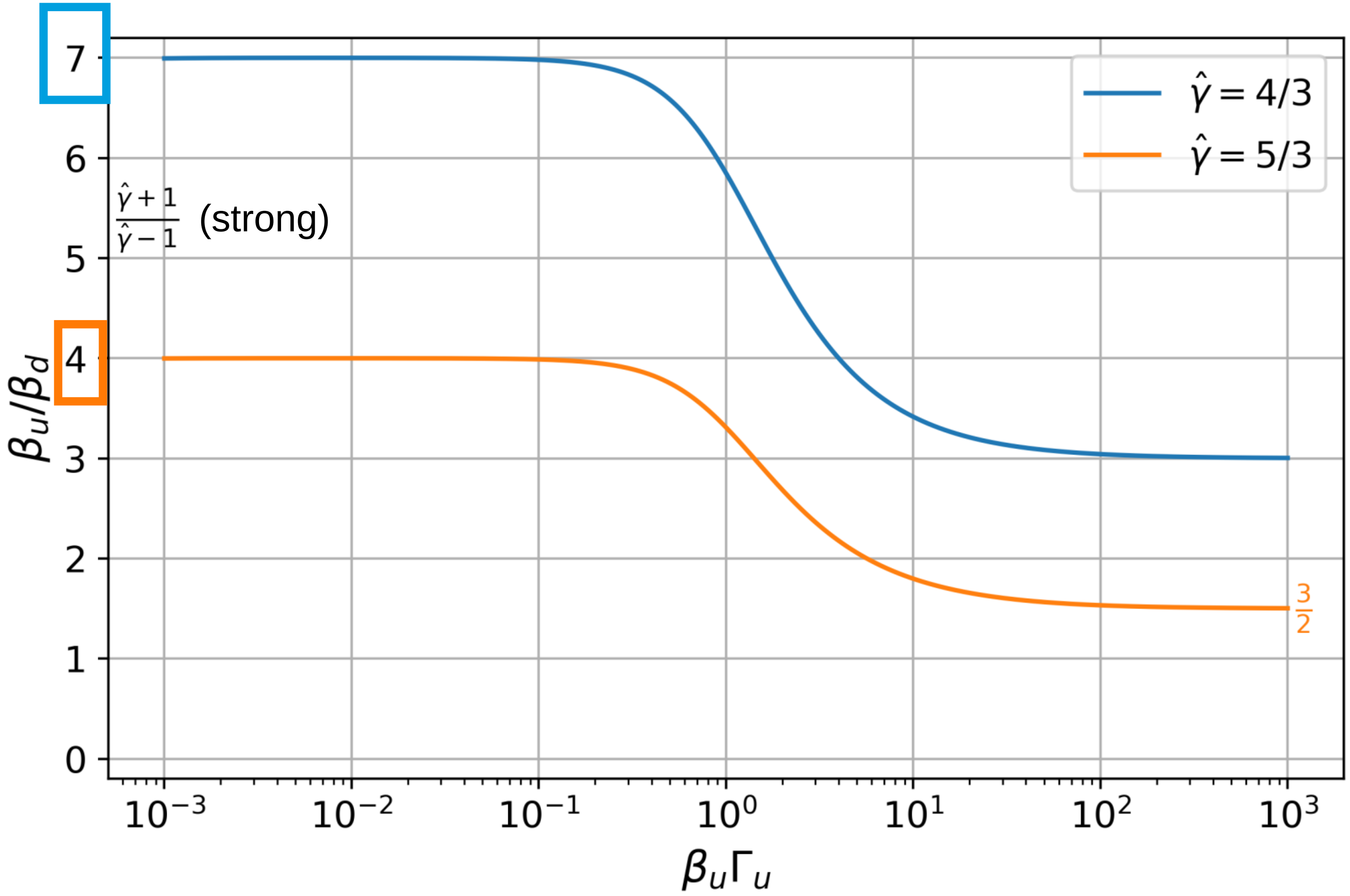




non-rel.: $\frac{\beta_u}{\beta_d} \approx \frac{\widehat{\gamma}+1}{\widehat{\gamma}-1}$

Deceleration



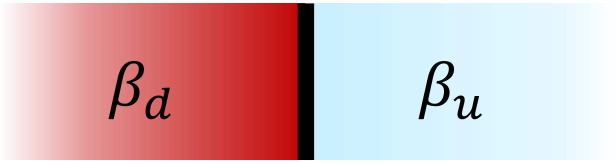


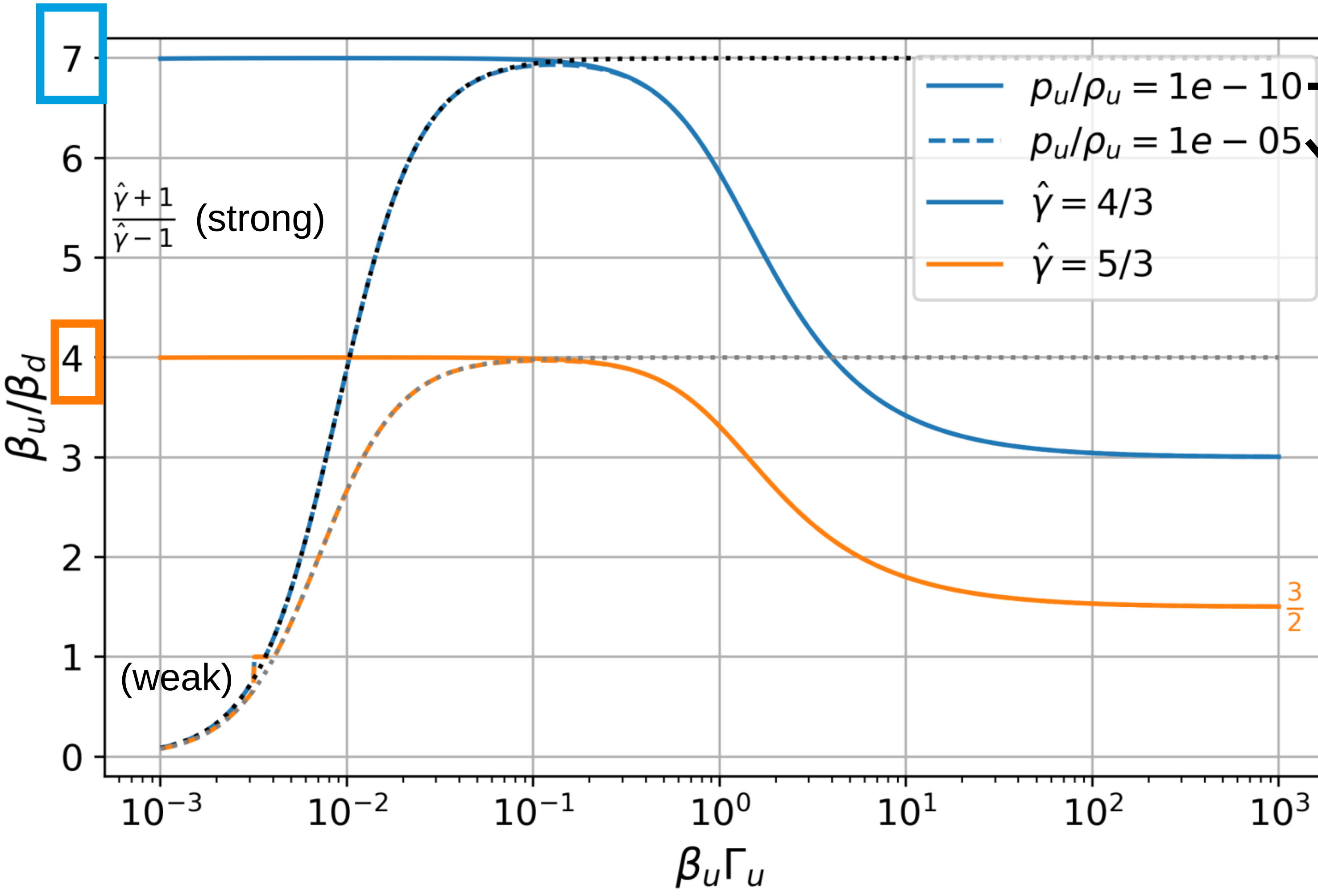
relativistic:

$$\frac{\beta_u}{\beta_d} \approx \frac{1}{\hat{\gamma}-1} = \begin{cases} 3 & \text{for } \hat{\gamma} = 4/3 \\ 1.5 & \text{for } \hat{\gamma} = 5/3 \end{cases}$$

non-rel.: $\frac{\beta_u}{\beta_d} \approx \frac{\hat{\gamma}+1}{\hat{\gamma}-1}$

Deceleration





$\mathcal{M} \gg 1$ for all $\beta_u\Gamma_u > 10^{-3}$

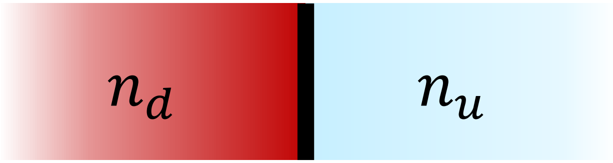
$\mathcal{M} \approx 1$ for $\beta_u\Gamma_u \approx \beta_u > \left(\frac{\rho_u}{p_u}\right)^{1/2}$

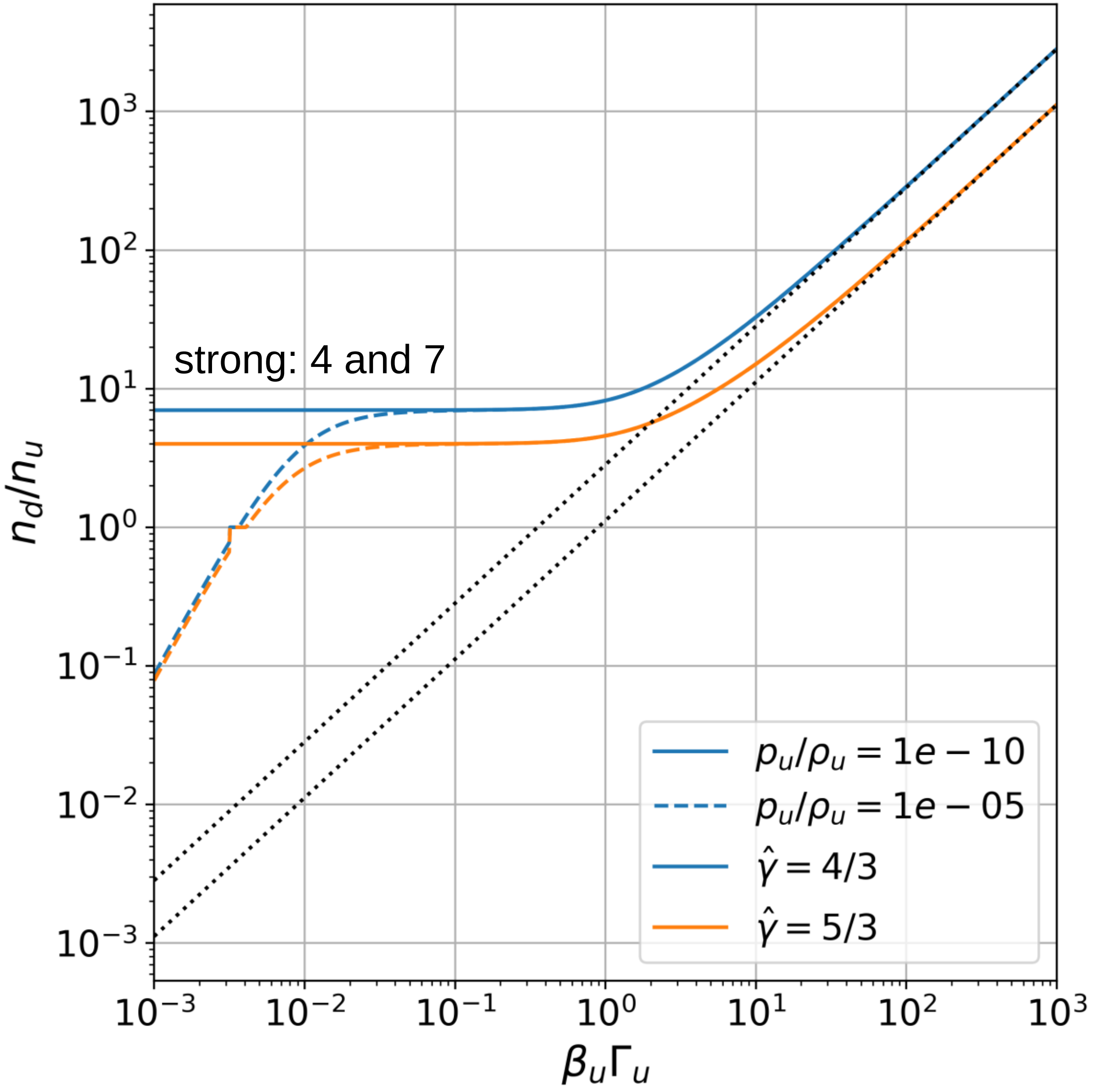
relativistic:

$$\frac{\beta_u}{\beta_d} \approx \frac{1}{\hat{\gamma}-1} = \begin{cases} 3 & \text{for } \hat{\gamma} = 4/3 \\ 1.5 & \text{for } \hat{\gamma} = 5/3 \end{cases}$$

non-rel.: $$\frac{\beta_u}{\beta_d} = \frac{\hat{\gamma}+1}{\hat{\gamma}-1+\frac{2}{\mathcal{M}^2}} = \frac{\hat{\gamma}+1}{\hat{\gamma}-1}\,\frac{1}{1+\frac{2\hat{\gamma}}{\hat{\gamma}-1}\,\frac{\rho_u}{p_u}\,\beta_u^2}$$

Number density - compression



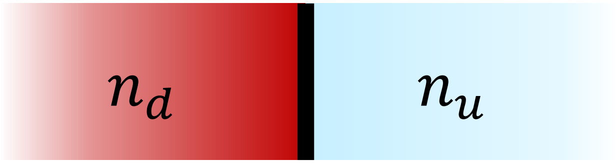

strong: 4 and 7

non-relativistic regime:

constant ratio of n_u and n_d
→ constant compression efficiency

$$\frac{n_d}{n_u} = \frac{\beta_u}{\beta_d} = \frac{\hat{\gamma}-1+\frac{2}{\mathcal{M}^2}}{\hat{\gamma}+1}$$

Number density - compression



non-relativistic regime:
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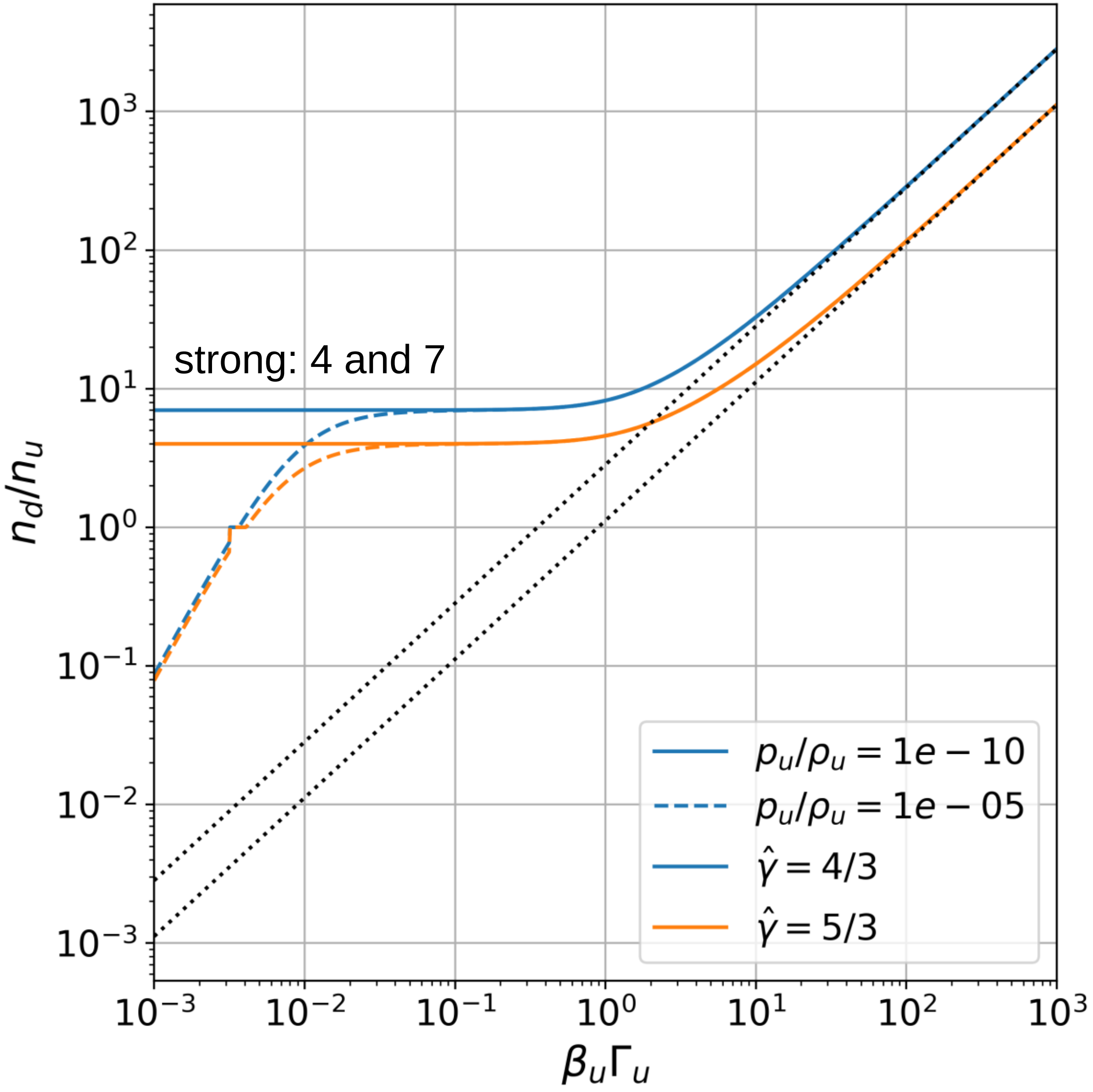


relativistic regime:
compression scales with Γ_u
→ arbitrary high compression

$$\frac{n_d}{n_u} = \frac{\overset{1}{\beta_u}\Gamma_u}{\underset{\text{const}}{\boxed{\beta_d\Gamma_d}}} \propto \Gamma_u$$

$$\beta_d\Gamma_d \approx \frac{\hat{\gamma}-1}{\sqrt{\hat{\gamma}(2-\hat{\gamma})}} = \frac{\hat{\gamma}-1}{\hat{\gamma}}\frac{\Gamma_u}{\Gamma_{ud}}$$

$\frac{n_d}{n_u} = \frac{\hat{\gamma}}{\hat{\gamma}-1}\Gamma_{\text{ud}}$ BM76, KZ14

Efficiency of pressure conversion

$$\varepsilon_{\mathrm{th}} = \frac{p_{\mathrm{d}}}{w_{\mathrm{u}}\beta^2\Gamma^2}$$

$$\boxed{[\![\beta^2\Gamma^2 w + p]\!] = 0}$$

$$\rightarrow p + \beta^2\Gamma^2 \frac{\hat{\gamma}}{\hat{\gamma}-1} p + \beta^2\Gamma^2 \rho$$

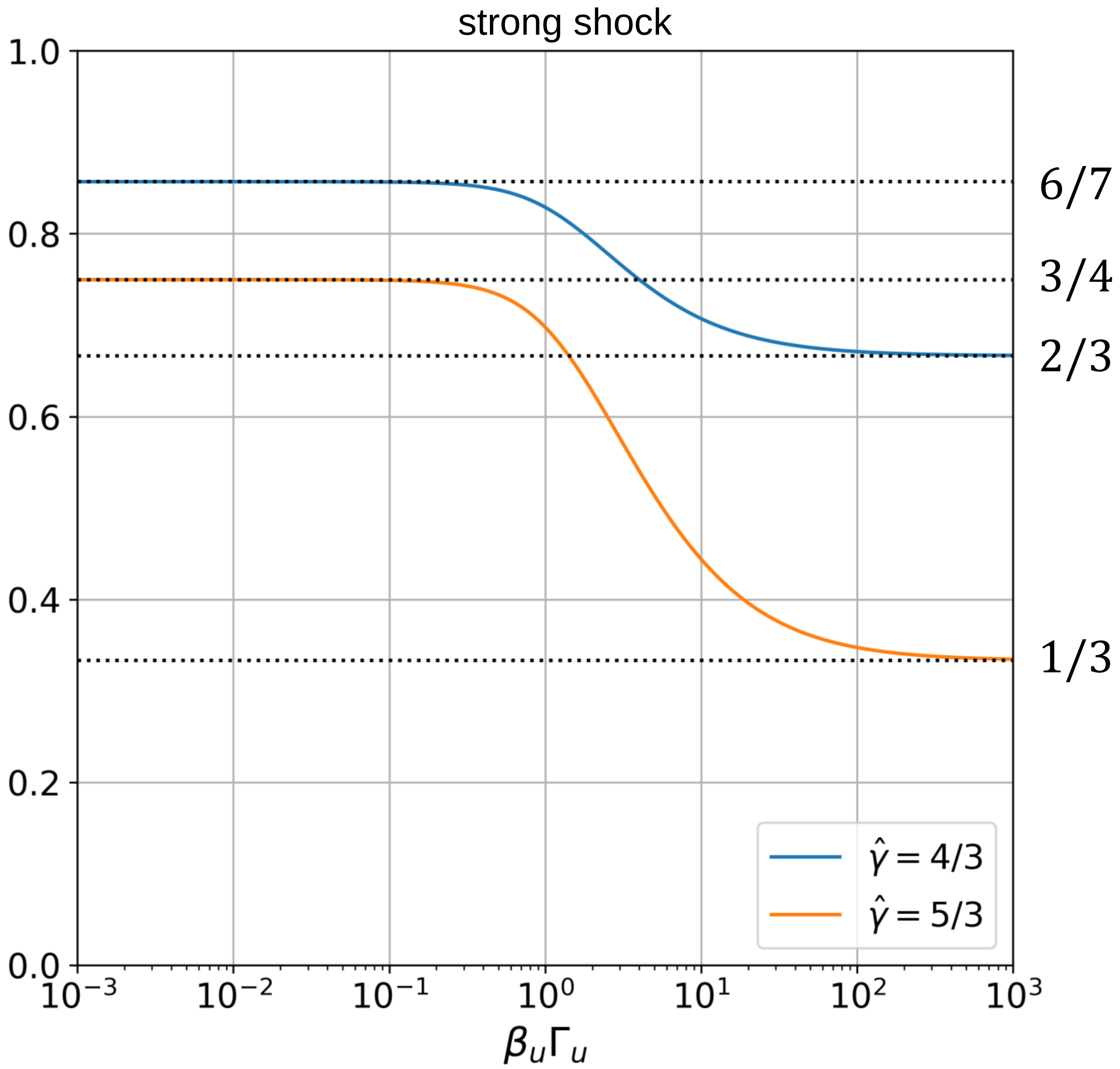
ram pressure

thermal pressure

consider ratio: $\dfrac{\text{outgoing thermal pressure}}{\text{incoming ram pressure}}$

→ **heating efficiency**

$$\varepsilon_X = \frac{p_X}{w_{\mathrm{u}}\beta^2\Gamma^2}$$



Efficiency of energy conversion

$$\boxed{[\![\beta\Gamma^2 w]\!] = 0} - \boxed{[\![n\Gamma\beta]\!] = 0}$$

$$\rightarrow \beta\Gamma^2 \frac{\hat{\gamma}}{\hat{\gamma}-1} p + \beta\Gamma(\Gamma - 1)\rho$$

thermal energy density flux

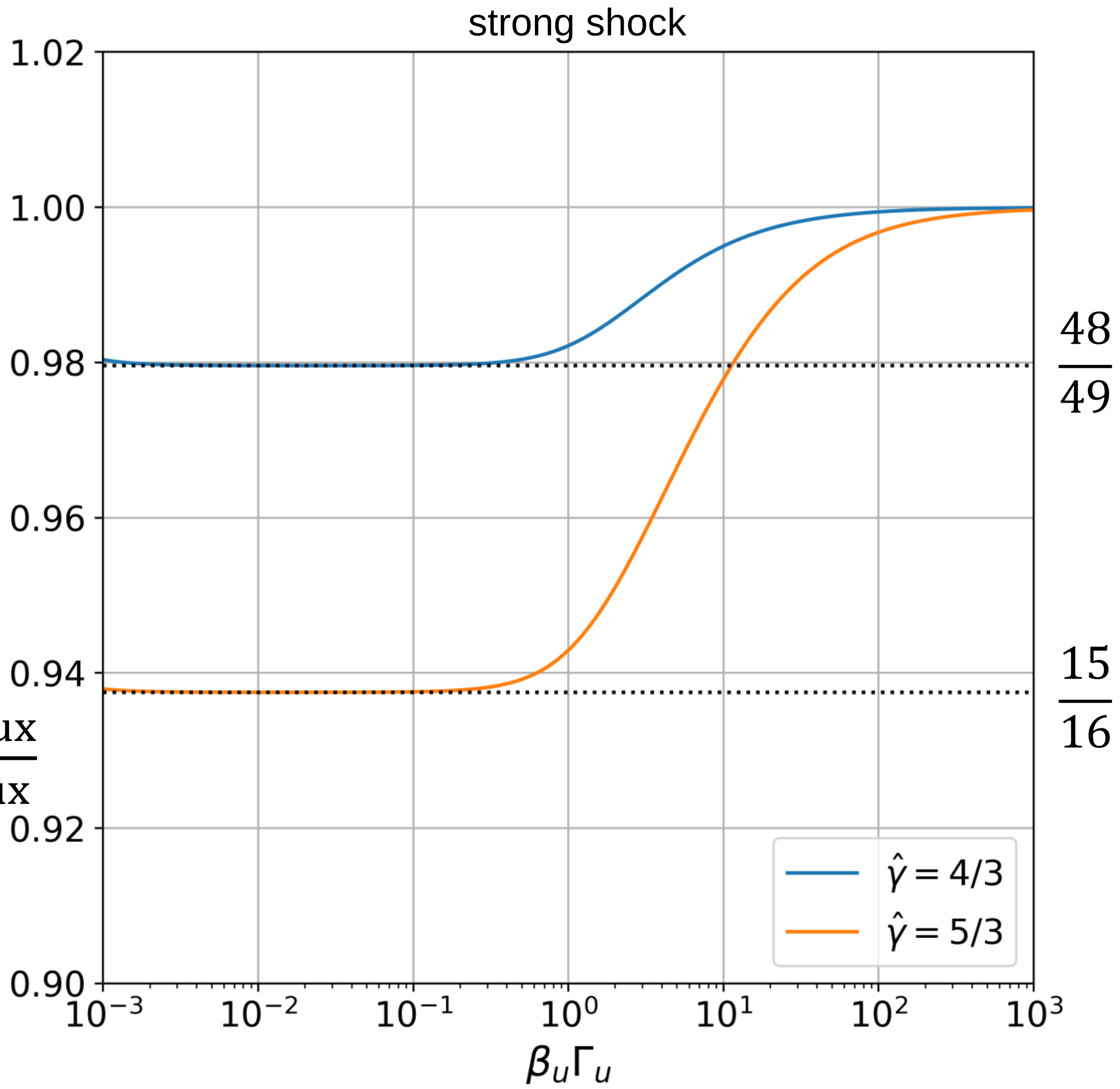
kinetic energy density flux

consider ratio: $\frac{\text{outgoing thermal energy density flux}}{\text{incoming kinetic energy density flux}}$

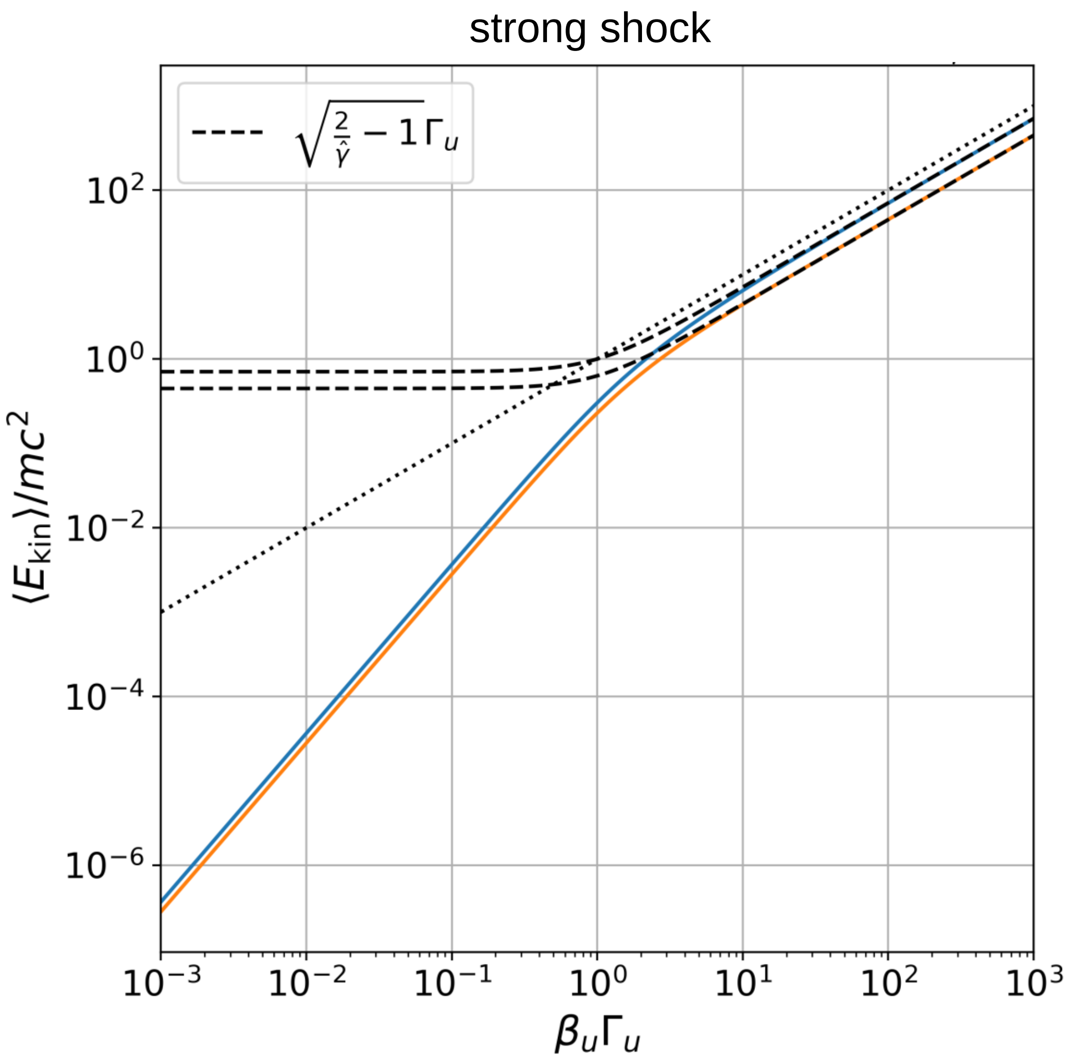
→ very close to 1

→ **thermal energy density (strong):**

$$e_d \propto \Gamma_u(\Gamma_\mathrm{u} - 1)\rho_u \underset{\text{rel.}}{\propto} \Gamma_u^2 \rho_u$$



Kinetic energy per particle (downstream rest frame)



$e = \frac{p}{\hat{\gamma}-1} + \rho$

$$\langle E_{kin}\rangle = \frac{e_d}{n_d} - mc^2$$

$$\rightarrow \quad \frac{\langle E_{kin}\rangle}{mc^2} = \frac{1}{\hat{\gamma}-1}\,\frac{\beta_d\Gamma_d}{\beta_u\Gamma_u}\,\frac{p_d}{p_u}\,\frac{p_u}{\rho_u}$$

$$\approx \underbrace{\sqrt{\frac{2}{\hat{\gamma}}-1}\,\Gamma_u}_{\Gamma_{\mathrm{ud}} = (1-\beta_u\beta_d)\Gamma_u\Gamma_d} \text{ for } \Gamma_u \gg 1$$

cf. Uhm12

→ relativistic shocks produce relativistic (thermal) particles!

→ pair creation?

Where is this description too simple?

- In the derivation we assumed that these values are defined infinitesimally close to the shock ($\epsilon \rightarrow 0$)

  → often assumed to hold within an entire homogeneous blast wave

- this simple 1D picture neglects any kind of turbulence

- feedback of non-thermal particles: We love to use shocks as sources of non-thermal particles (e.g. via diffusive shock acceleration)

  → the particles heat up the upstream medium (see e.g. Caprioli et al. 2020)

  → these shocks are collisionless, so we need to add magnetic fields to diffuse particles

Summary on relativistic shocks

$$\boxed{[\![n\Gamma\beta]\!] = 0}$$

$$\boxed{[\![\beta^2\Gamma^2 w + p]\!] = 0}$$

$$\boxed{[\![w\Gamma^2\beta]\!] = 0}$$

- all shocks convert incoming bulk kinetic energy into heating
- downstream speed always transrelativistic (in the shock rest frame)
- density compression by Γ_{u}
- thermal energy density increase by factor Γ_u^2

References

- A. Taub 1948, Relativistic Rankine-Hugoniot Equations
- L. Landau & E. Lifshitz 1987, Fluid Mechanics
- B. Schutz 2009, A First Course in General Relativity, Second Edition
- L. Rezolla & O. Zanotti 2013, Relativistic Hydrodynamics

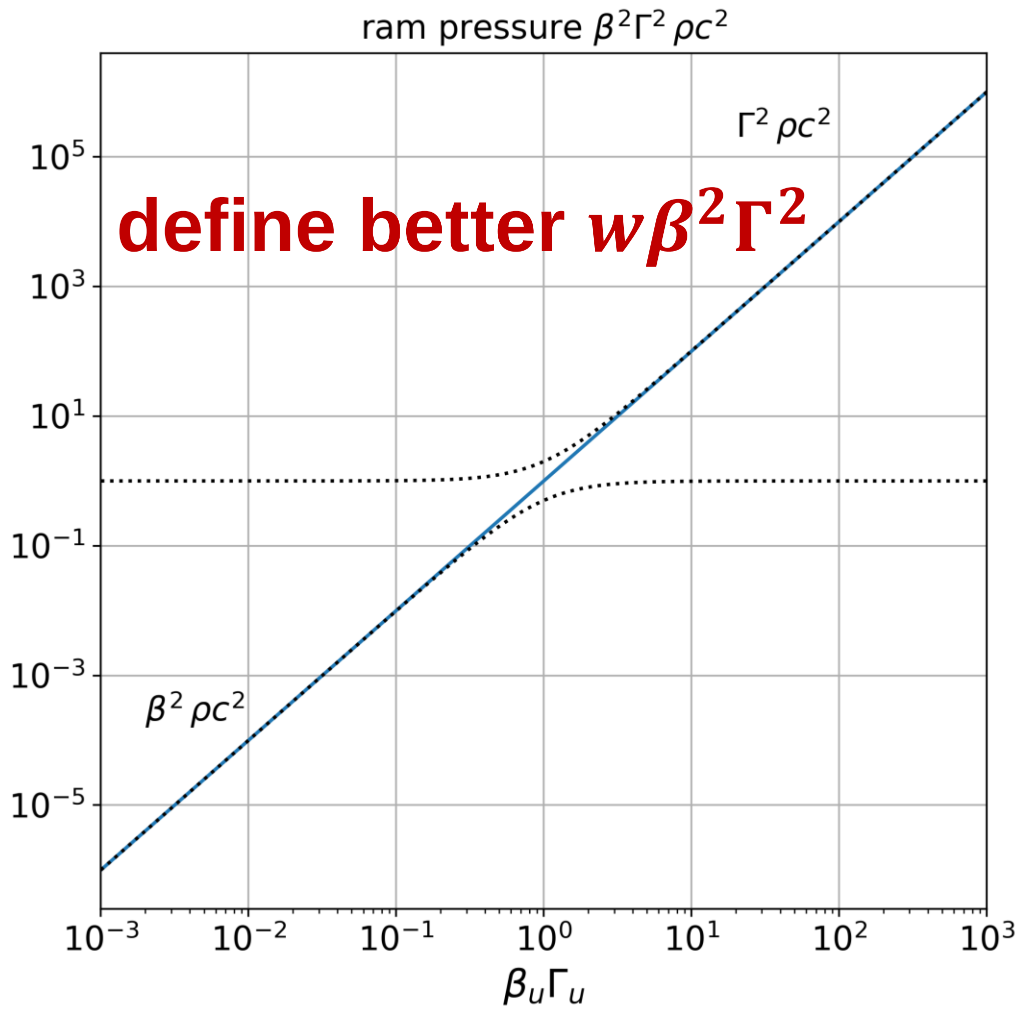
- Many thanks to Andrew Taylor, Vasu Shaw, Pavlo Plotko for enlightening discussions!

Energy Momentum Tensor in Rest frame of fluid

$$T^{\mu\nu} = w u^{\mu} u^{\nu} + p_{\mathrm{th}} g^{\mu\nu}$$

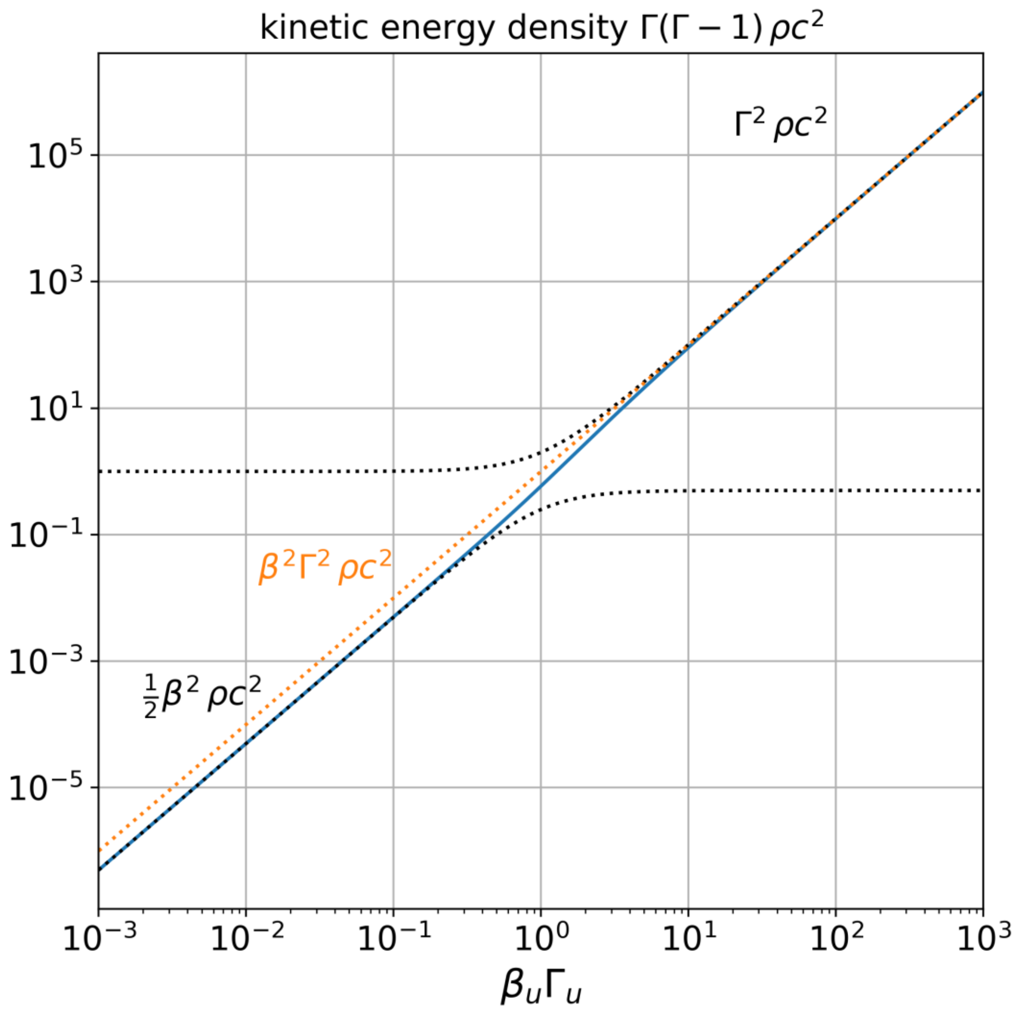
$$\rightarrow T^{\mu\nu} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Relativistic ram pressure & kinetic energy density



ram pressure $\beta^2\Gamma^2\,\rho c^2$

define better $w\beta^2\Gamma^2$



kinetic energy density $\Gamma(\Gamma-1)\,\rho c^2$