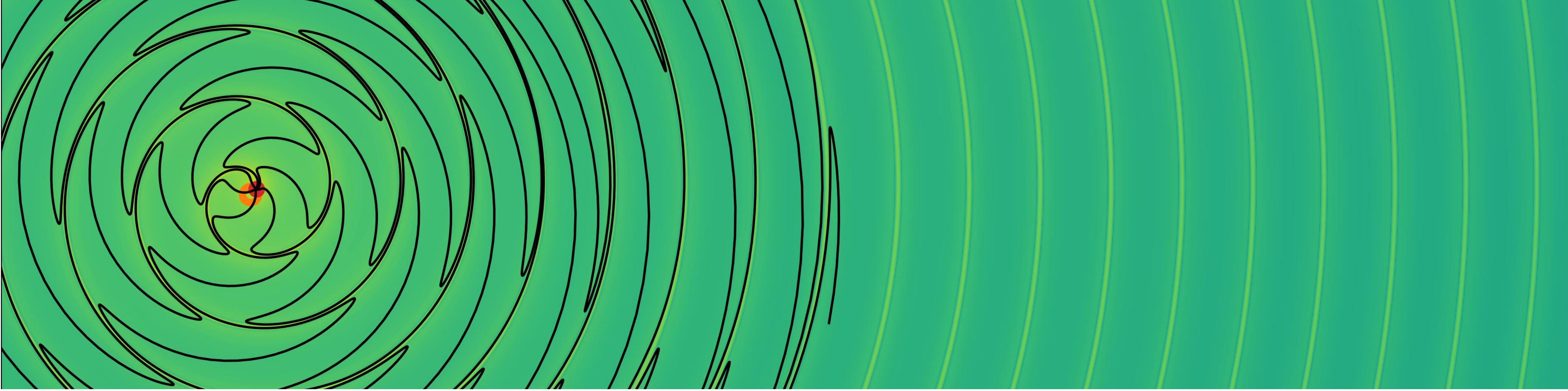


Weizsäcker-Williams approach

Marc Klinger

MMS Science Club 26.01.2023

HELMHOLTZ WEIZMANN RESEARCH SCHOOL MULTIMESSENGER ASTRONOMY

HELMHOLTZ

DESY.

What is it all about?

- radiation of a point charge moving at relativistic non-const. speed

  → radiation often produced by moving charges, e.g.

    - synchrotron
    - bremsstrahlung
    - Cherenkov radiation

  → derivation of spectral distribution often quite mathematically complex

  → not very intuitive results ($\int d\ln x \, K_{\frac{5}{3}}(x)$)

  → tend to skip derivations, use end results, loose track of assumptions and approximations made

- Weizsäcker-Williams method as simple approximation based geometrical considerations with intuitive picture

Weizsäcker-Williams Approach

- charged particles at relativistic speeds carry a “pancake” field around them (in obs. frame)

  → can be seen as a **cloud of virtual photons**

- if particle gets deviated from it’s trajectory ($\dot{\vec{v}} \neq 0$), virtual photons are shaken off after a **formation time**

- simple intuition that works for many radiation processes

- geometry/trajectory defines radiated spectrum

  → WW approach gives good approximations without heavy mathematical machinery

Wikipedia

C.F. Weizsäcker

https://mathshistory.st-andrews.ac.uk/Biographies/Williams_Evan/

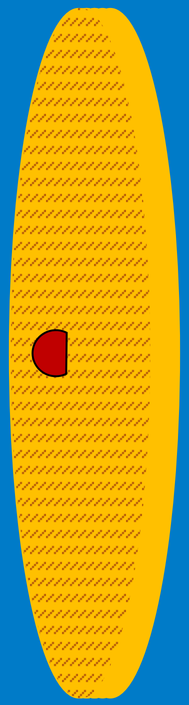
E.J. Williams

Weizsäcker, C. F. v. Ausstrahlung bei Stößen sehr schneller Elektronen. *Zeitschrift für Physik* **88**, 612–625 (1934).

Williams, E. J. CORRELATION OF CERTAIN COLLISION PROBLEMS WITH RADIATION THEORY. Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. 13, No. 4 (1935)

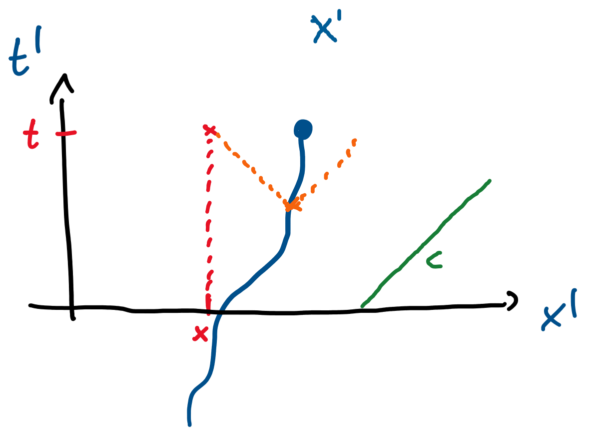
Zolotorev, M. S. & McDonald, K. T. Classical Radiation Processes in the Weizsacker-Williams Approximation. (2000).

Idea of virtual photon cloud



1. Potentials
2. Fields
3. Energy spectrum

Green's function of a 4D-point charge



- Maxwell's equations in vacuum (Lorenz Gauge)

→ $\Box\phi(\vec{x},t) = \rho(\vec{x},t)$ and $\Box\vec{A}(\vec{x},t) = \vec{j}(\vec{x},t)$ with $\Box = \frac{1}{c^2}\partial_t^2 - \Delta$

→ Green's function = solution for point charge in space and time

→ $G_\pm(\vec{x}-\vec{x}',t'-t) = \Theta(\mp(t'-t))\frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left((t'-t) \pm \frac{|\vec{x}'-\vec{x}|}{c}\right)$

limit in time
-: retarded
+: advanced

point charge from electrostatics

fixes speed of information to speed of light
→ radial EM wave

→ solution for any charge/current distribution:

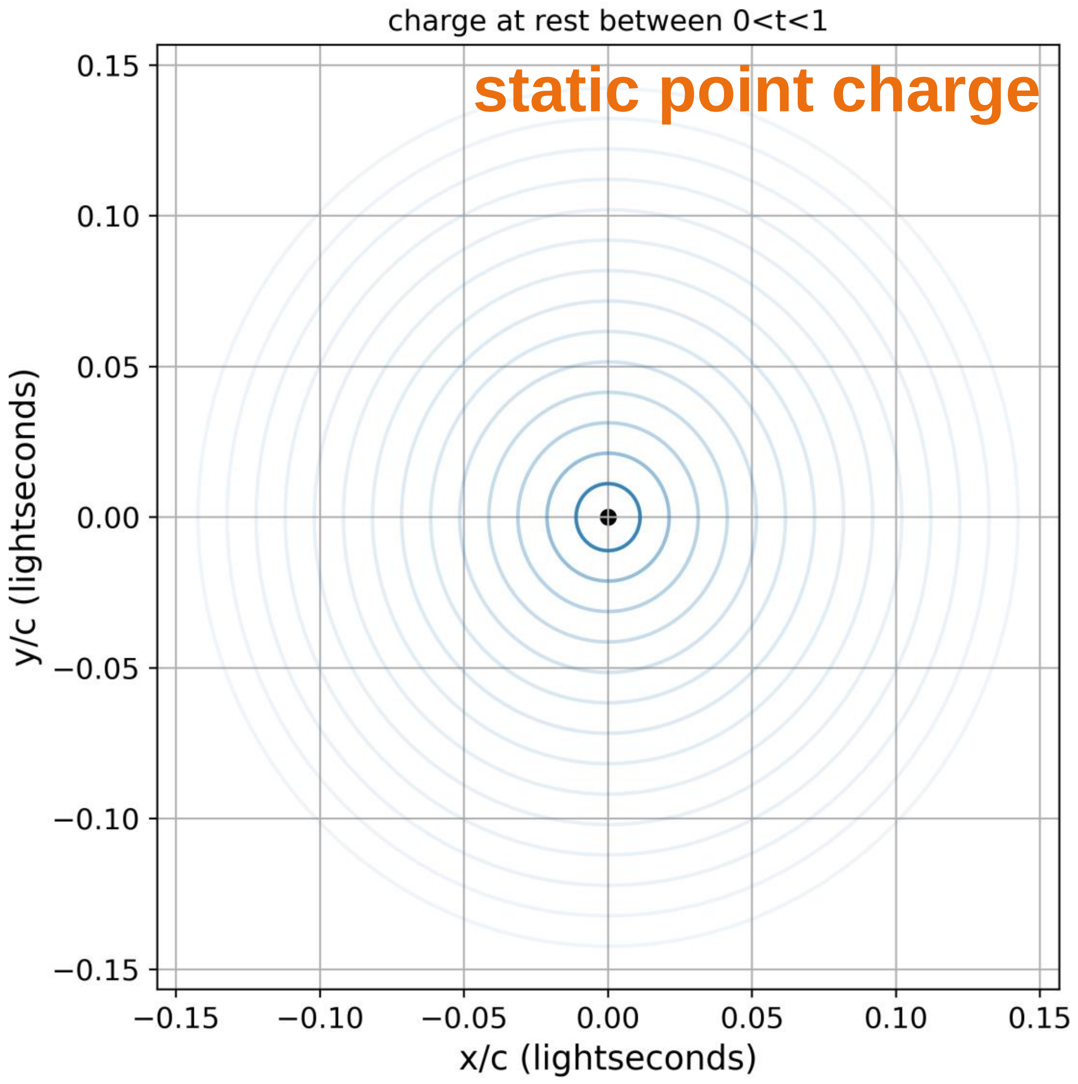
→ $\phi_\pm(\vec{x},t) = \int \mathrm{d}t'\mathrm{d}V'\, G_\pm(\vec{x}'-\vec{x},t'-t)\,\rho(\vec{x}',t') = \int \mathrm{d}V' \frac{1}{|\vec{x}-\vec{x}'|}\rho\left(\vec{x}',t \mp \frac{|\vec{x}'-\vec{x}|}{c}\right)$

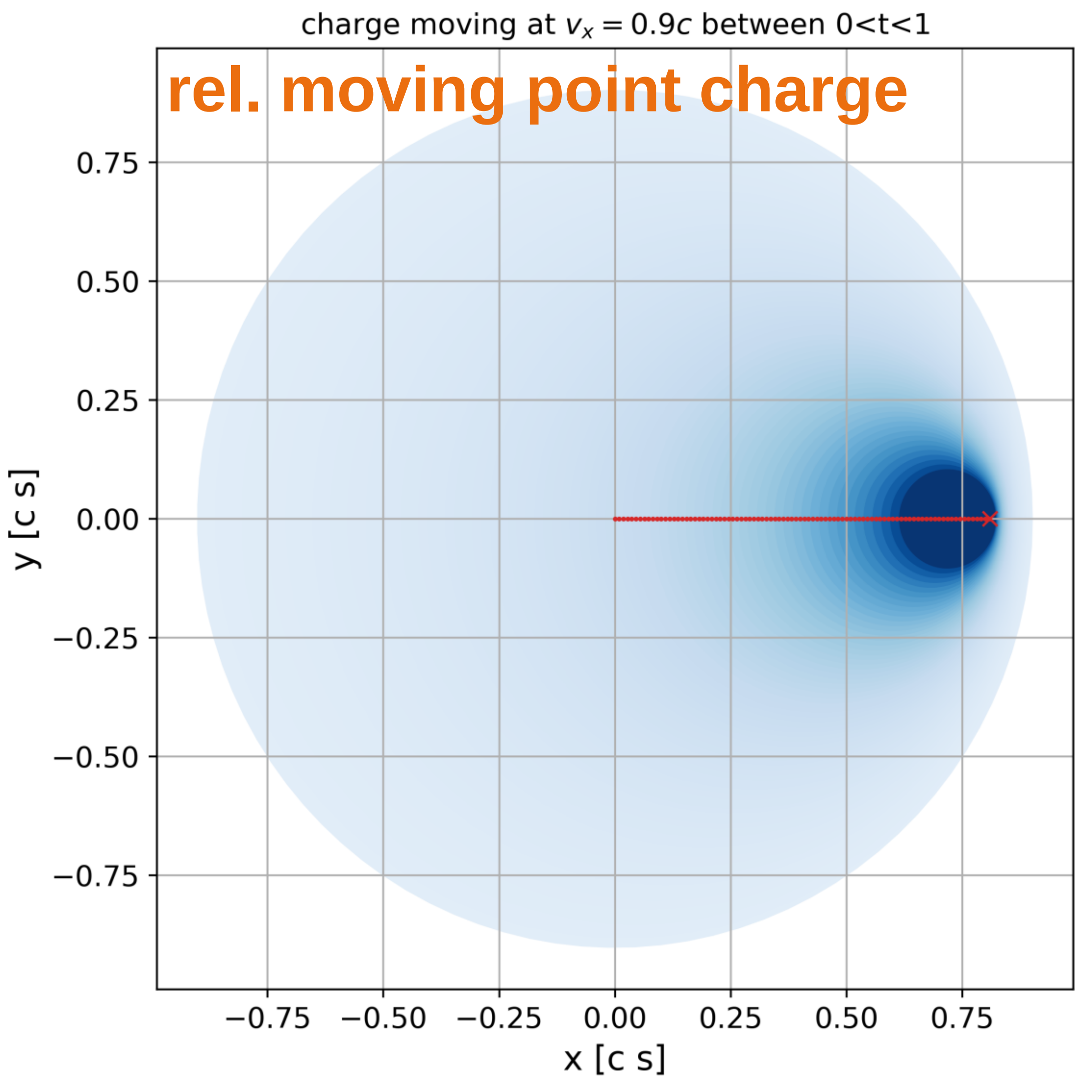
same for vector potential $\vec{A}$ with current distribution $\vec{j}$

Green's function propagates charge distribution in time/space

see e.g. Jackson, Classical Electrodynamics

Potentials visually





Electromagnetic fields of a point charge

- Fields correspond to gradients of potentials (mix both, ϕ and $\vec{A}$)

  $\rightarrow \vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$

- for point charge with trajectory $\vec{R}(t)$ and $\vec{\beta}(t) = \partial_t \vec{R}(t)/c$:

  $\rightarrow \rho(\vec{x}, t) = q\, \delta^3\left(\vec{x} - \vec{R}(t)\right)$ and $\vec{j}(\vec{x}, t) = q\vec{\beta}c\delta^3\left(\vec{x} - \vec{R}(t)\right)$

$$\rightarrow \vec{E} = q\left(\underbrace{\frac{\gamma^2}{R^2}\frac{\vec{n}-\vec{\beta}}{\left(1-\vec{n}\cdot\vec{\beta}\right)^3}}_{\text{Coulomb field (with Doppler factor)}} + \underbrace{\frac{\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]}{R\left(1-\vec{n}\cdot\vec{\beta}\right)^3}}_{\text{acceleration term} \rightarrow \text{radiation field}}\right)_{t_r}, \quad \vec{B} = \vec{n} \times \vec{E}$$

with $\vec{n} = \vec{x} - \vec{R}(t)$

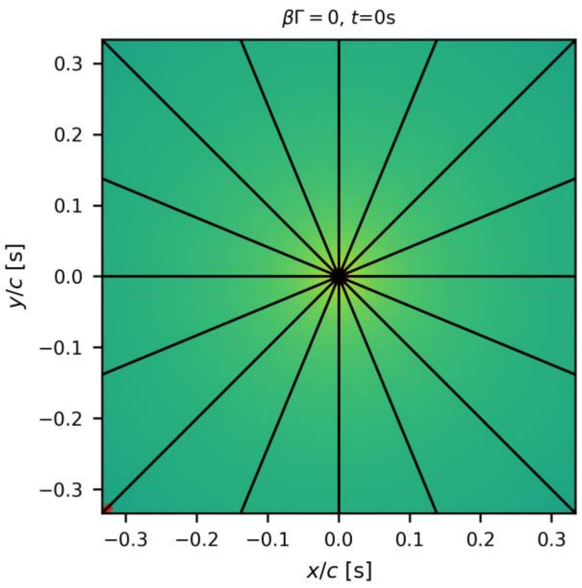
evaluated at retarded time t_r, where $t - t_r = \frac{\left|\vec{R}(t_r) - \vec{x}\right|}{c}$

see e.g. Jackson, Classical Electrodynamics

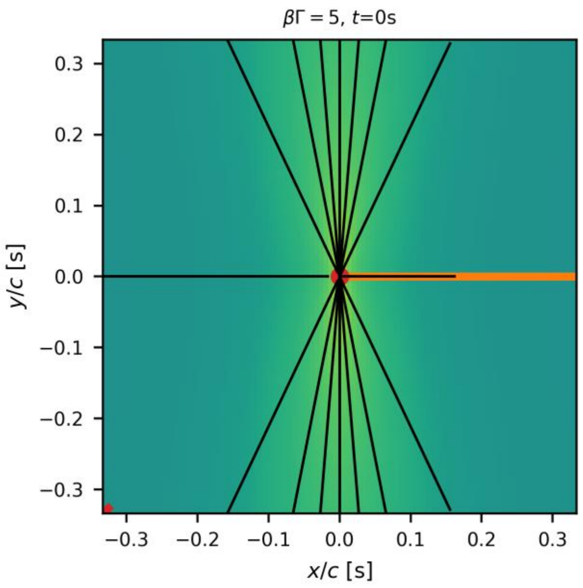
Visual fields: Field lines for relativistic point charge

- Field lines = tangential to electric field vector

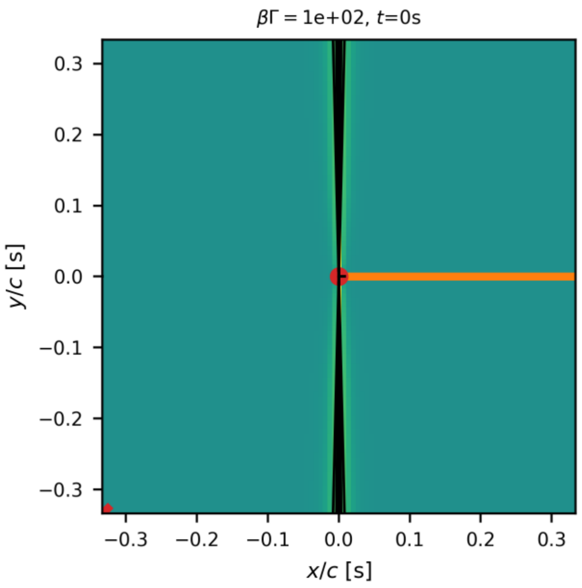
at rest



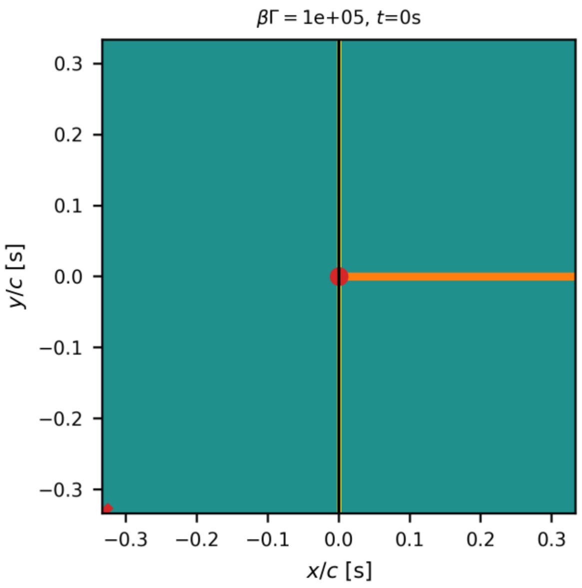
few MeV electron



50 MeV electron



500 GeV electron

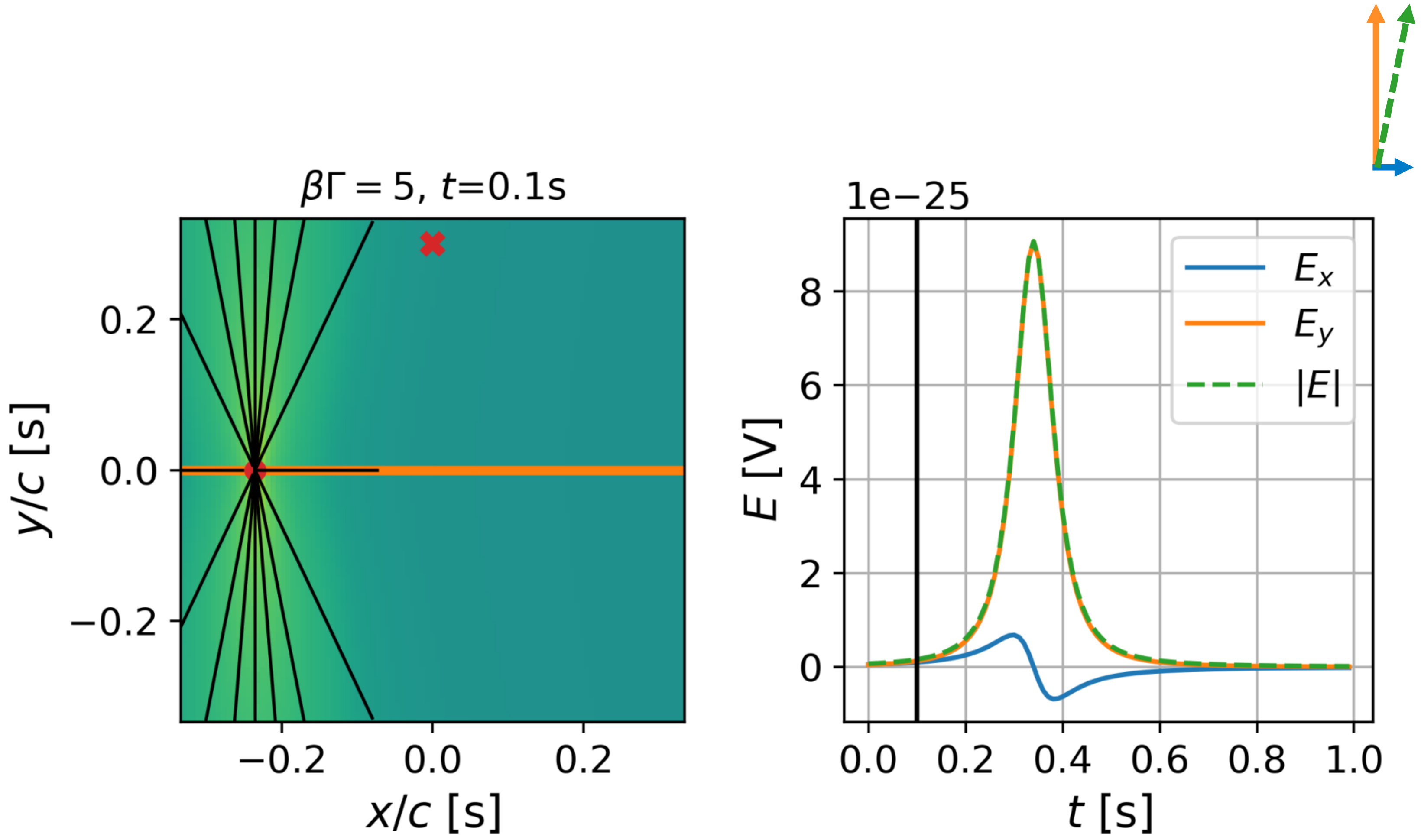


field lines are amplified perpendicular to direction of motion
→ field becomes basically transverse

python module from Matthew Filipovic: https://github.com/MatthewFilipovich/moving-point-charges
see also Filipovich, M. J. & Hughes, S. *American Journal of Physics* **89**, 482–489 (2021).

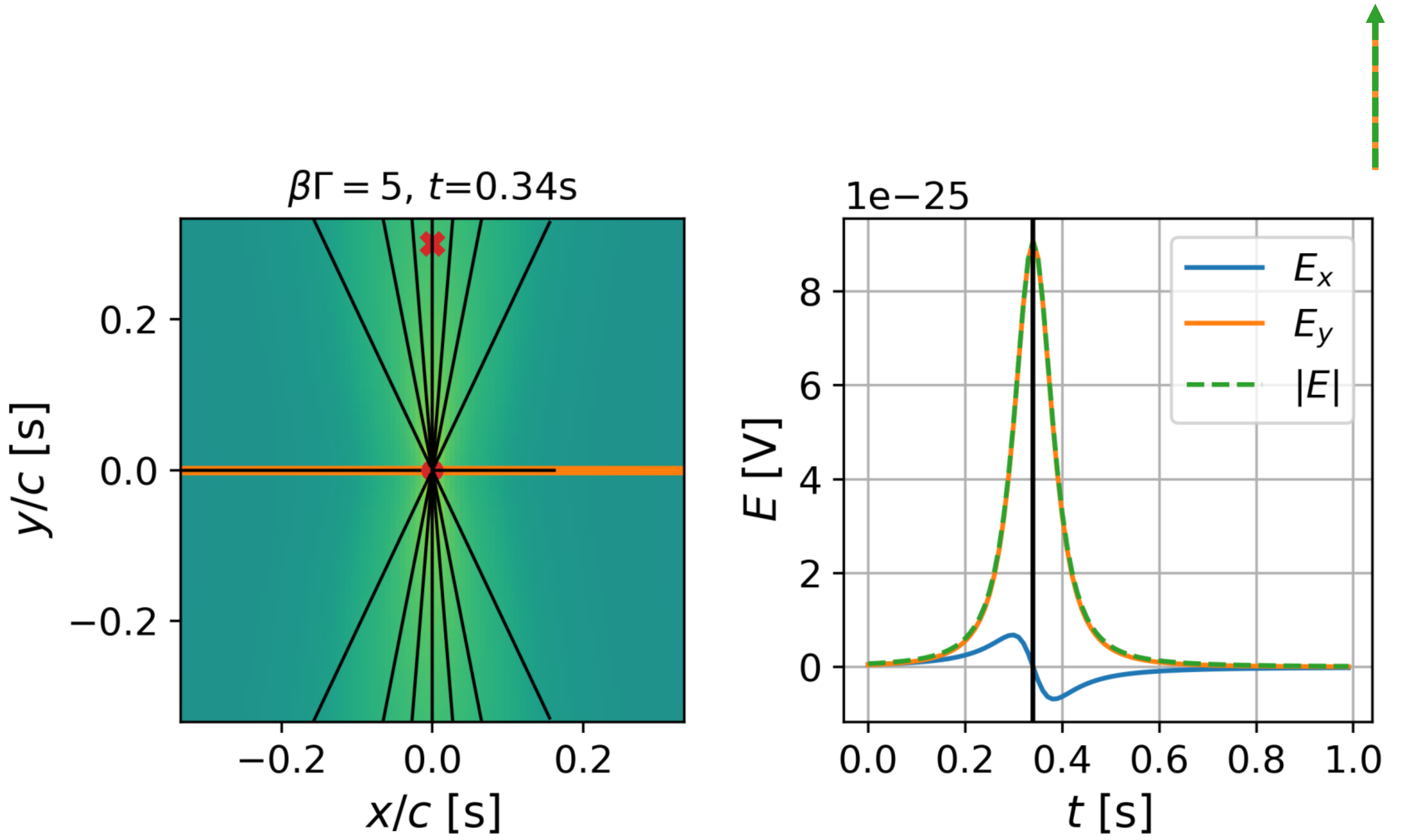
What does an observer see?

- field at distance basically an EM wave traveling with photon



What does an observer see?

- field at distance basically an EM wave traveling with photon



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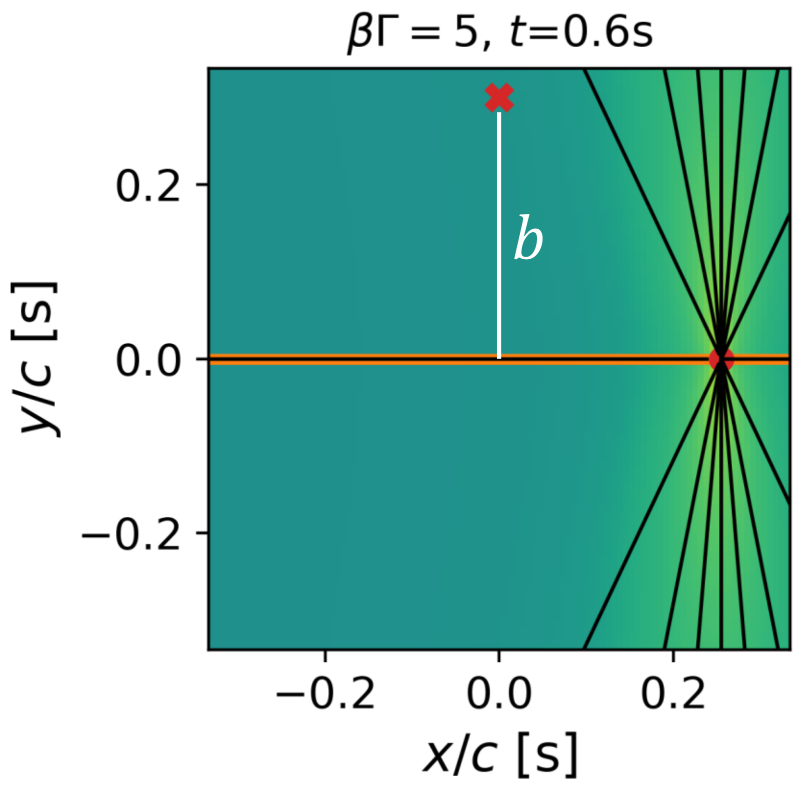
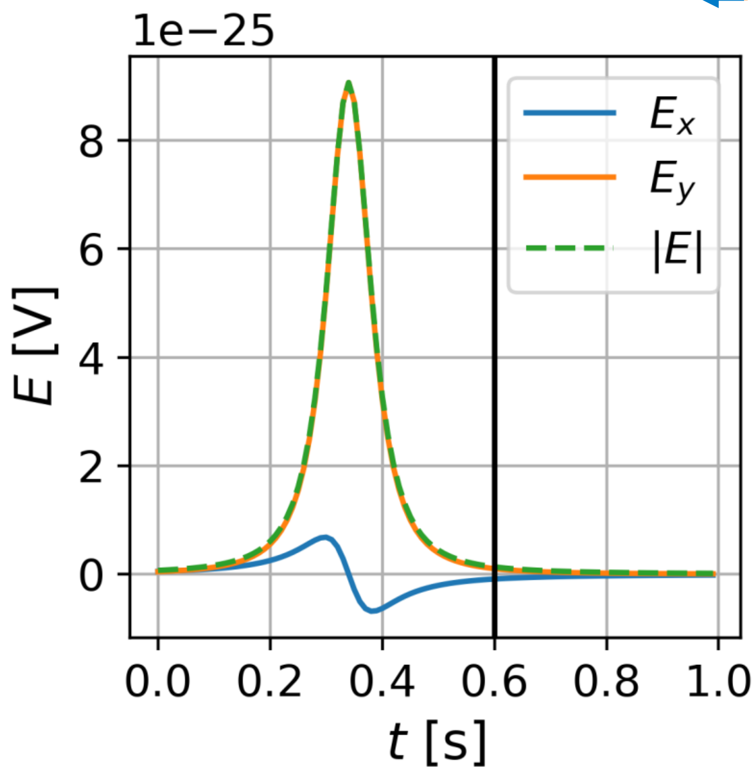




mangetic field: $\vec{B} = \vec{n} \times \vec{E}$

alternatively:
boost Coulomb field to observer frame

$$E_x(t) = \frac{q}{b^2}\frac{\beta\gamma ct}{b}\left(1 + \left(\frac{\gamma ct}{b}\right)^2\right)^{-3/2}$$

$$E_y(t) = \frac{q\gamma}{b^2}\left(1 + \left(\frac{\gamma ct}{b}\right)^2\right)^{-3/2}$$

Pulse Properties

$$E_x(t) = \frac{q}{b^2}\frac{\beta\gamma c t}{b}\left(1 + \left(\frac{\gamma c t}{b}\right)^2\right)^{-3/2}$$

$$E_y(t) = \frac{q\gamma}{b^2}\left(1 + \left(\frac{\gamma c t}{b}\right)^2\right)^{-3/2}$$

(technically $t \to t - t_0$)

- at their max. values

  $\to E_x \sim \frac{q}{b^2} \to$ negligible

  $\to E_y \sim \frac{q\gamma}{b^2} \to$ plane wave

- pulse significant for $\Delta t \approx \frac{b}{\gamma c}$

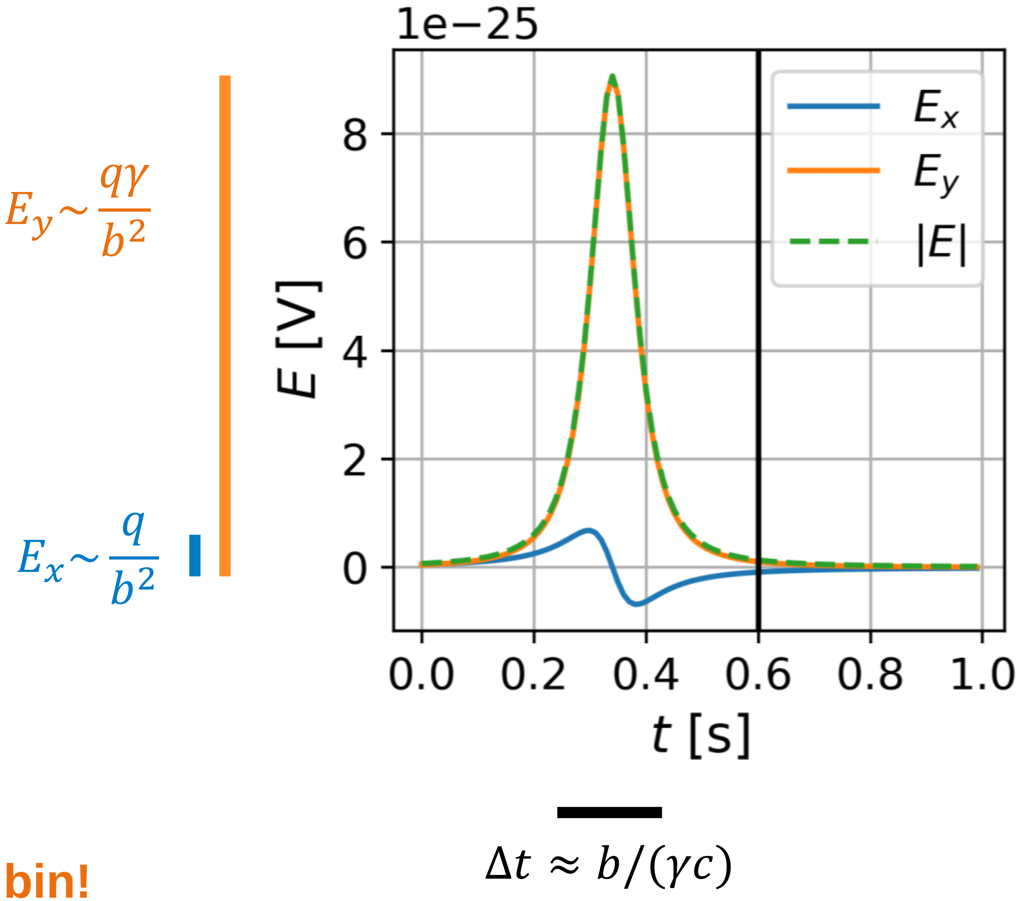
  $\to$ dominant frequency $\nu \sim \frac{1}{\Delta t} = \frac{\gamma c}{b}$

- energy $U \sim E^2 V \sim \frac{e^2\gamma^2}{b^4} \; b^2 \frac{b}{\gamma} \sim \frac{e^2\gamma}{b}$

- for peaky energy spectrum

  $\to \frac{E\mathrm{d}N}{\mathrm{d}E} = \frac{\mathrm{d}N}{\mathrm{d}\ln E} \sim \frac{U}{h\nu} \sim \frac{e^2}{\hbar c} = \alpha$

  $\to$ constant amount $\sim\alpha$ of photons per log. energy bin!



Spectrum of the virtual photon cloud

Fermi, E. Über die Theorie des Stoßes zwischen Atomen und elektrisch geladenen Teilchen. *Zeitschrift für Physik* **29**, 315–327 (1924).

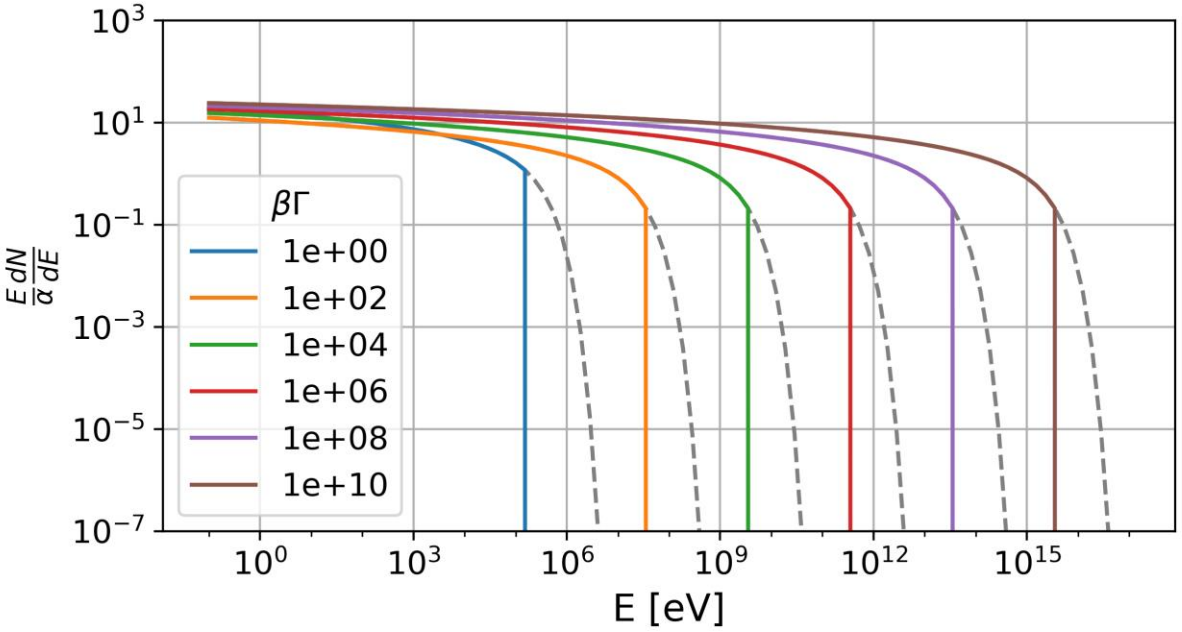
- Fourier transform and integrate for all distances (b) above b_{min}

$$\rightarrow \frac{1}{\alpha}\frac{\mathrm{d}N}{\mathrm{dln}E} = \frac{2}{\pi\beta^2}\left[xK_0(x)K_1(x) - \frac{x^2}{2}\left(K_1^2(x) - K_0^2(x)\right)\right] \quad \text{with } x = \frac{Eb_{min}}{\beta\gamma\hbar c}$$

cf. Rybicki & Lightman eq. 4.74b

$$\rightarrow \frac{1}{\alpha}\frac{\mathrm{d}N}{\mathrm{dln}E} \approx \theta\left(E \leq \frac{\gamma c\hbar}{b_{min}}\right) \quad \text{for } \beta \approx 1 \quad \text{(neglecting logarithmic term)}$$

photon distribution in entire cloud, electron

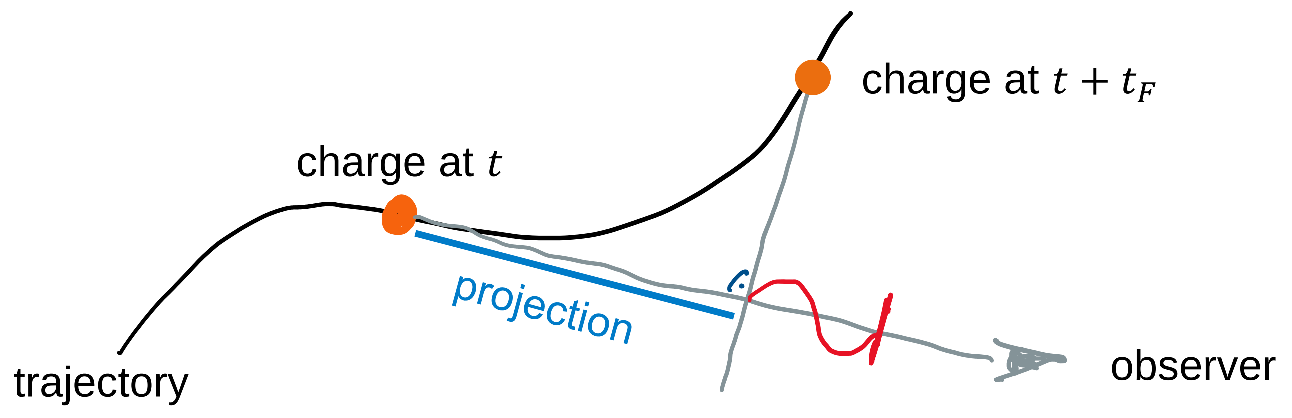


absolute minimum on b from Compton wavelength of particle (max energy $E_{max} \lesssim \frac{\gamma c\hbar}{b_{min}} \sim \gamma mc^2$)

→ idea of WW:
these virtual photons are ready to be liberated

Formation time
or
how to free the virtual photons

How to free photons?



*“The **formation length (time)** is the distance (time) the electron travels while a radiated wave advances one wavelength ahead of the projection of the electron’s motion onto the direction of observation.”*

Zolotorev, M. S. & McDonald, K. T. Classical Radiation Processes in the Weizsacker-Williams Approximation. (2000).

Where does this come from?

Schwinger, J. On the Classical Radiation of Accelerated Electrons. *Phys. Rev.* **75**, 1912–1925 (1949)

- radiation = charge works against its own old (retarded) fields
- can calculate the power: $\frac{\mathrm{d}E}{\mathrm{d}t}(t) = -\int \mathrm{d}V\, \vec{j}(\vec{x},t) \cdot \vec{E}_{\mathrm{ret}}(\vec{x},t)$
- consider only radiated part, point charge, Fourier transform

$$\rightarrow E\frac{\mathrm{d}N}{\mathrm{d}E\mathrm{d}t} = \frac{\alpha}{2}\int_{-\infty}^{0}\frac{\mathrm{d}\tau}{\tau_E}\left(1-\vec{\beta}\cdot\vec{\beta}'\right)\left[\frac{\sin\left(\omega_E\left[\tau-\frac{|\vec{R}'-\vec{R}|}{c}\right]\right)}{|\vec{R}'-\vec{R}|/c}-\frac{\sin\left(\omega_E\left[\tau+\frac{|\vec{R}'-\vec{R}|}{c}\right]\right)}{|\vec{R}'-\vec{R}|/c}\right]$$

velocity dispersion (Doppler factor)

step function with formation time

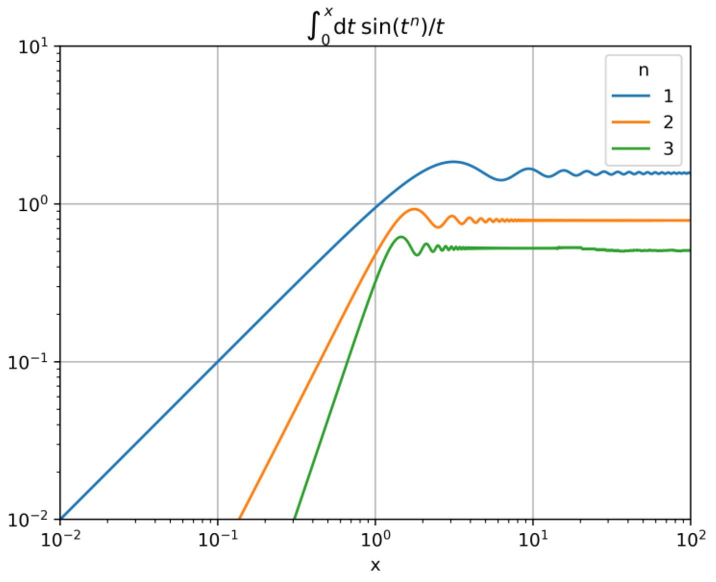
cut-off: at max. energy cancels step term

$$\tau_E = \frac{h}{E}$$

$$\omega_E = \frac{2\pi}{\tau_E} = \frac{E}{\hbar}$$

$$\vec{R}' - \vec{R} = \vec{R}(\tau+t) - \vec{R}(t) \sim \left(\frac{\tau}{t_F}\right)^n$$

$$\int dx \frac{\sin x^n}{x} \longrightarrow$$



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- consider only radiated part, point charge, Fourier transform

WW approximation

$$\rightarrow E\frac{\mathrm{d}N}{\mathrm{d}E\mathrm{d}t} = \frac{\alpha}{2}\int_{-\infty}^{0}\frac{\mathrm{d}\tau}{\tau_E}\left(1-\vec{\beta}\cdot\vec{\beta}'\right)\left[\frac{\sin\left(\omega_E\left[\tau-\frac{|\vec{R}'-\vec{R}|}{c}\right]\right)}{|\vec{R}'-\vec{R}|/c}-\frac{\sin\left(\omega_E\left[\tau+\frac{|\vec{R}'-\vec{R}|}{c}\right]\right)}{|\vec{R}'-\vec{R}|/c}\right] \approx \boxed{\frac{\alpha}{t_F}\theta(t>t_F)\theta(E<E_{max})}$$

velocity dispersion (Doppler factor)

step function with formation time

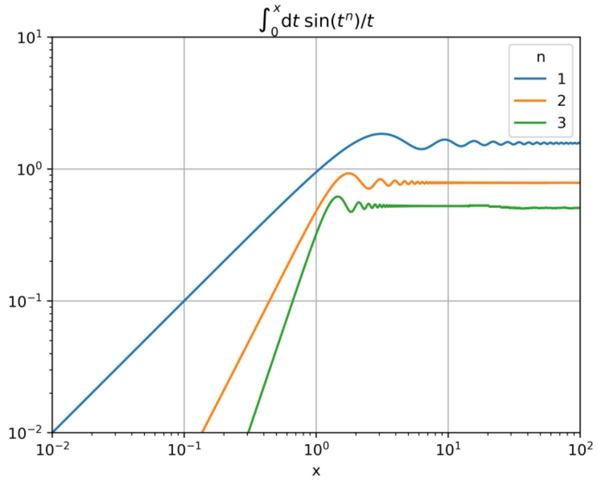
cut-off: at max. Energy cancels step term

$$\tau_E = \frac{h}{E}$$

$$\omega_E = \frac{2\pi}{\tau_E} = \frac{E}{\hbar}$$

$$\vec{R}'-\vec{R} = \vec{R}(\tau+t)-\vec{R}(t) \sim \left(\frac{\tau}{t_F}\right)^n$$

$$\int dx\,\frac{\sin x^n}{x} \longrightarrow$$



WW summary

- Radiation spectrum:

$$\frac{\mathrm{d}N}{\mathrm{d}\ln E\;\mathrm{d}t} \approx \frac{\alpha}{t_F(E)} e^{-\frac{E}{E_c}} \theta\big(t \geq t_F(E)\big)$$

- need to work out:

  $\rightarrow E_c$ critical energy from e.g. b_{min}

  $\rightarrow t_F$ formation time from geometry of trajectory

- radiation beamed into $1/\gamma$ cone for rel. particle

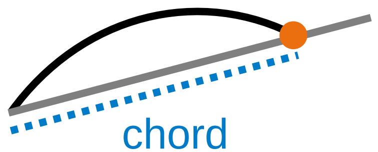
  $\rightarrow$ only very small part of curved trajectory matters

Examples

1) Synchrotron
2) Cherenkov

1) Synchrotron in the WW approach

- pulse duration for small angle interval: $\Delta t_{obs} \sim \mathfrak{D}\,\frac{R_L}{c}\,\Delta\theta \sim \frac{1}{\gamma^2}\frac{R_L}{c}\frac{1}{\gamma}$
  - Doppler factor $\mathfrak{D} = 1 - \vec{\beta}\cdot\vec{n} \approx \frac{1}{\gamma^2}$, Larmor radius $R_L = \frac{\beta\gamma m_e c^2}{eB}$, angular width $\Delta\theta \sim \frac{1}{\gamma}$



- critical energy: $E_{max} \sim \gamma^3\,\frac{\hbar c}{R_L} = \gamma^2\,\frac{B}{B_c}\,m_e c^2$

- formation time: $\lambda = ct_F - \Delta_{chord} \approx ct_F\left[\frac{1}{2\gamma^2} + \frac{1}{24}\left(\frac{ct_F}{R_L}\right)^2\right]$

→ negligible curvature effects: $\frac{1}{\gamma^2} \gg \left(\frac{L_F}{r}\right)^2 = \theta_F^2$ → first term dominates, no radiation

→ dominant curvature effects: $\frac{1}{\gamma^2} \ll \theta_F^2$ → second term dominates, $t_F \approx \frac{(24R_L^2\lambda)^{\frac{1}{3}}}{c} \sim h^{\frac{1}{3}}\left(\frac{R_L}{c}\right)^{\frac{2}{3}} E^{-\frac{1}{3}}$

- $\frac{\mathrm{d}N}{\mathrm{d}\ln E\, dt} \approx \frac{\alpha}{t_F(E)}\, e^{-\frac{E}{E_{max}}} \propto E^{\frac{1}{3}}\, e^{-\frac{E}{E_{max}}}$

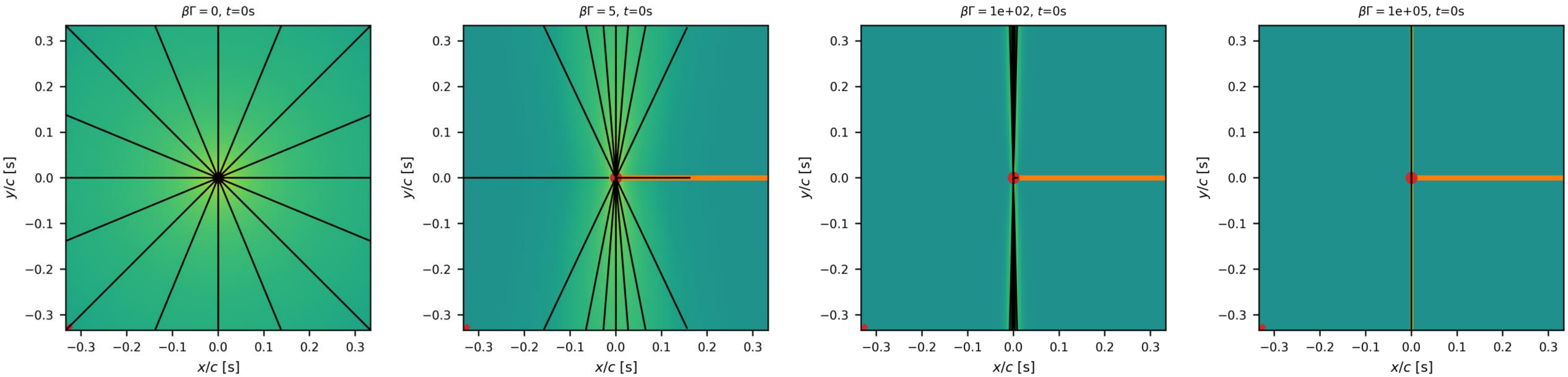
Field lines for constant motion

at rest

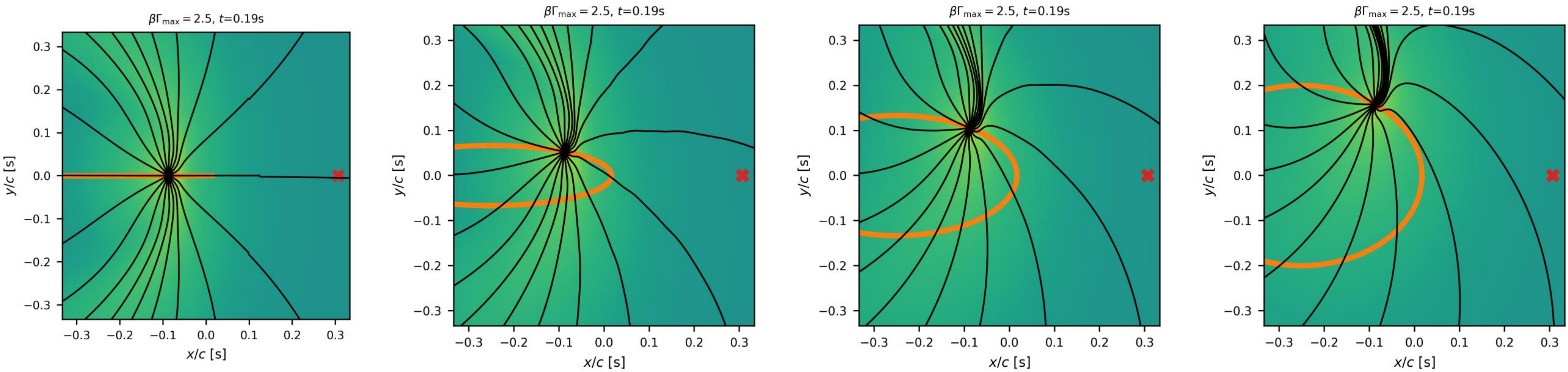
few MeV electron

50 MeV electron

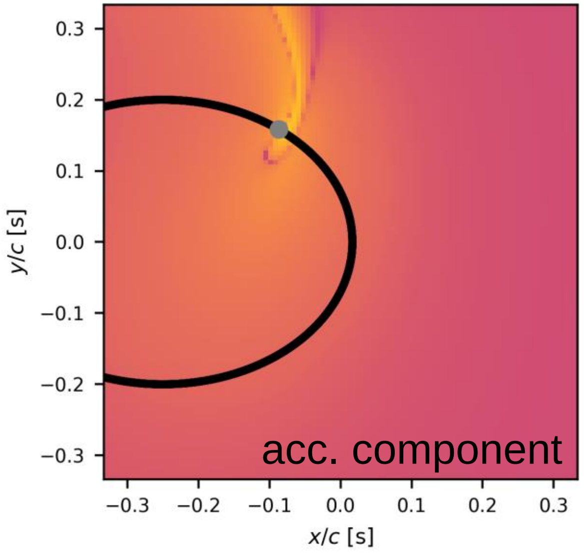
500 GeV electron



Field lines for curved trajectory

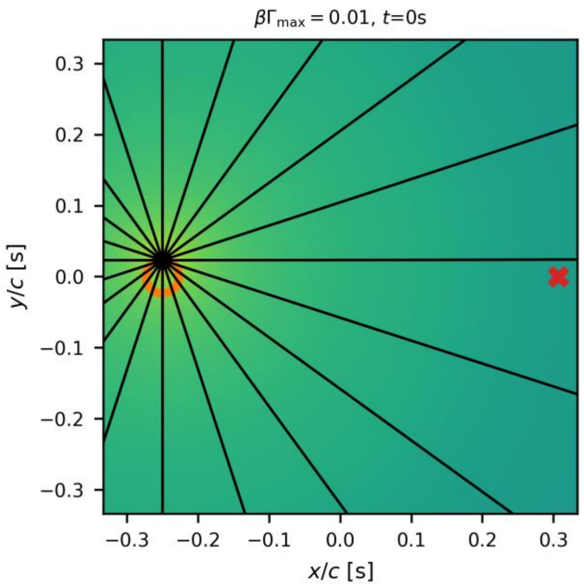


- Intuition for field lines
  - → radially (relativistically compressed) blown out at c along particles history
  - → curvature creates radially outwards compressed field lines

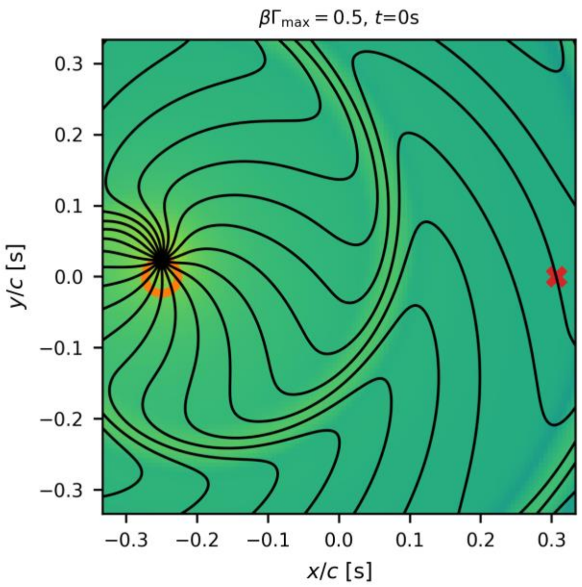


Visual fields: Field lines for circle

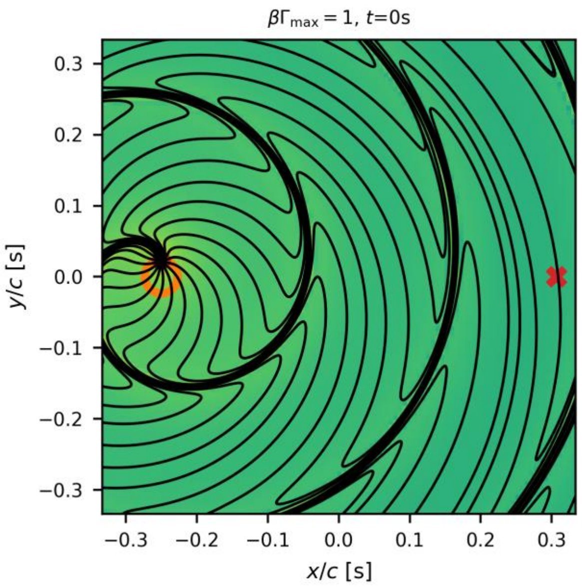
non-rel.



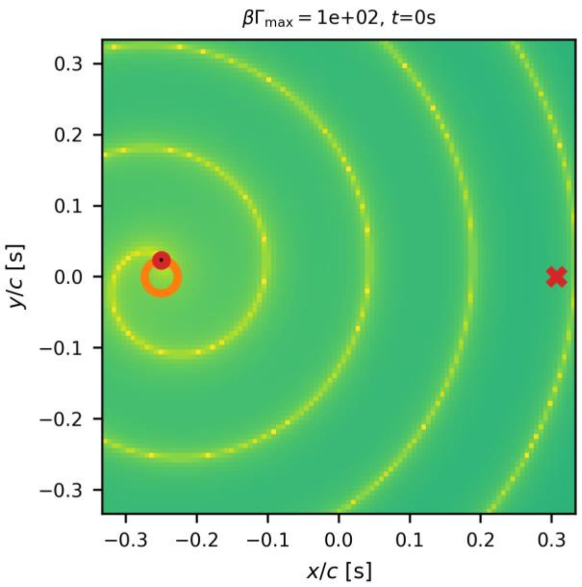
mildly rel. electron



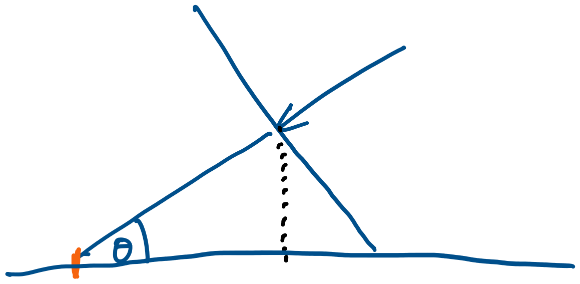
0.5 MeV electron



50 MeV electron



2) Cherenkov radiation



- particle faster than speed of light in medium

  → outruns virtual photons

  → characteristic angle $\cos\theta_C = \frac{1}{\beta n}$

- $\lambda = \beta c t_F - \frac{c}{n} t_F \cos\theta_C = \beta c t_F \sin^2\theta_C$

  $\rightarrow t_F = \frac{\lambda}{\beta c \sin^2\theta_C} \approx \frac{h}{\sin^2\theta_C E}$

  $\rightarrow \frac{\mathrm{d}N}{\mathrm{dln}E\, dt} \approx \frac{\alpha}{t_F(E)} = \frac{\alpha}{h}\sin^2\theta_C\ E$

- Franck-Tamm: $\frac{\mathrm{d}E}{\mathrm{d}\omega\,\mathrm{d}x} = \frac{\hbar\,\mathrm{d}N}{\mathrm{dln}E\, c\mathrm{d}t} \approx \frac{\hbar\alpha}{c t_F(E)} = \frac{e^2}{2\pi c^2}\omega\left(1 - \left(\frac{1}{\beta n}\right)^2\right)$

WW summary

- Radiation spectrum:

$$\frac{\mathrm{d}N}{\mathrm{dln}E\ \mathrm{d}t} \approx \frac{\alpha}{t_F(E)} e^{-\frac{E}{E_c}} \theta\big(t \geq t_F(E)\big)$$

- need to work out:

  →E_c critical energy from e.g. b_{min}

  →t_F formation time from geometry of trajectory

- radiation beamed into $1/\gamma$ cone for rel. particle

  → only very small part of curved trajectory matters

Bibliography

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- Rybicki, G. B. & Lightman, A. P. *Radiative Processes in Astrophysics*. (Wiley, 2004).

Syn: Formation length

- $\lambda = ct_F - \overline{OP}$ and $L_F = r\theta_F$

  $\rightarrow \overline{OP} = \overline{OT}\, r\cos\frac{\theta_F}{2} = 2r\cos\frac{\theta_F}{2}\sin\frac{\theta_F}{2} = r\sin\theta_F$

- $\lambda = L_F\left(\frac{1}{\beta} - \frac{\sin\theta_F}{\theta_F}\right)$

- with $\frac{\sin\theta_F}{\theta_F} \approx 1 - \frac{\theta_F^2}{6}$ and $\frac{1}{\beta} \approx 1 + \frac{1}{2\gamma^2}$

- $\lambda = \frac{L_F}{2\gamma^2} + \frac{L_F^3}{6r^2} + \cdots$

- two regimes:

  $\rightarrow$ negligible curvature effects $\frac{1}{\gamma^2} \gg \left(\frac{L_F}{r}\right)^2 = \theta_F^2$: first term dominates

  $\rightarrow$ dominant curvature effects $\gamma^{-2} \ll \theta_F^2$: second term dominates